

# Composite bundles in Clifford algebras. Gravitation theory. Part I

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## Abstract

Based on a fact that complex Clifford algebras of even dimension are isomorphic to the matrix ones, we consider bundles in Clifford algebras whose structure group is a general linear group acting on a Clifford algebra by left multiplications, but not a group of its automorphisms. It is essential that such a Clifford algebra bundle contains spinor subbundles, and that it can be associated to a tangent bundle over a smooth manifold. This is just the case of gravitation theory. However, different these bundles need not be isomorphic. To characterize all of them, we follow the technique of composite bundles. In gravitation theory, this technique enables us to describe different types of spinor fields in the presence of general linear connections and under general covariant transformations.

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# 1 Introduction

In this work, we aim to describe spinor fields in gravitation theory in terms of bundles in Clifford algebras. A problem is that gauge symmetries of gravitation theory are general covariant transformations whereas spinor fields carry out representations of Spin groups which are two-fold covers of the pseudo-orthogonal ones.

In classical gauge theory, a case of matter fields which admit only a subgroup of a gauge group is characterized as a spontaneous symmetry breaking [4, 17, 22]. Spontaneous symmetry breaking is a quantum phenomenon, but it is characterized by a classical background Higgs field [17, 23]. In classical gauge theory on a principal bundle  $P \rightarrow X$  with a structure Lie group  $G$ , spontaneous symmetry breaking is defined as a reduction of this group to its closed (consequently, Lie) subgroup  $H$  (Definition 5.1). By virtue of the well-known Theorem 5.2, there is one-to-one correspondence between the  $H$ -principal subbundles  $P^h$  of  $P$  and the global sections of the quotient bundle  $\Sigma = P/H \rightarrow X$  (5.1) with a typical fibre  $G/H$ . These sections are treated as classical Higgs fields [4, 16, 22]. Matter fields possessing only exact symmetry group  $H$  are described in a pair with Higgs fields as sections of composite bundles  $Y \rightarrow \Sigma \rightarrow X$  [20, 22].

This is just the case of Dirac spinor fields in gravitation theory [4, 13, 18].

Theory of classical fields on a smooth manifold  $X$  admits a comprehensive mathematical formulation in the geometric terms of smooth fibre bundles over  $X$  [4, 19]. For instance, Yang – Mills gauge theory is theory of principal connections on a principal bundle  $P \rightarrow X$  with some structure Lie group  $G$ . Gauge gravitation theory is formulated in the terms of fibre bundles which belongs to the category of natural bundles [4, 18].

Studying gauge gravitation theory, one requires that it incorporates Einstein's General Relativity and, therefore, it should be based on Relativity and Equivalence Principles reformulated in the fibre bundle terms [6, 11]. As a consequence, gravitation theory has been formulated as gauge theory of general covariant transformations with a Lorentz reduced structure where a pseudo-Riemannian metric gravitational field is treated as the corresponding classical Higgs field [6, 11, 14, 15].

Relativity Principle states that gauge symmetries of classical gravitation theory are general covariant transformations. Fibre bundles possessing general covariant transformations constitute the category of so called natural bundles [4, 8, 25].

**Remark 1.1:** Let  $\pi : Y \rightarrow X$  be a smooth fibre bundle. Any automorphism  $(\Phi, f)$  of  $Y$ , by definition, is projected as  $\pi \circ \Phi = f \circ \pi$  onto a diffeomorphism  $f$  of its base  $X$ . The converse need not be true. A fibre bundle  $Y \rightarrow X$  is called the natural bundle if there exists a monomorphism

$$\text{Diff } X \ni f \rightarrow \tilde{f} \in \text{Aut } Y$$

of a group of diffeomorphisms of  $X$  to a group of bundle automorphisms of  $Y \rightarrow X$ . Automorphisms  $\tilde{f}$  are called general covariant transformations of  $Y$ .

The tangent bundle  $TX$  of  $X$  exemplifies a natural bundle. Any diffeomorphism  $f$  of  $X$  gives rise to the tangent automorphisms  $\tilde{f} = Tf$  of  $TX$  which is a general covariant transformation of  $TX$ . The associated principal bundle is a fibre bundle  $LX$  of linear frames in tangent spaces to  $X$  (Section 5.2). It also is a natural bundle. Moreover, all fibre bundles associated to  $LX$  are natural bundles.  $\square$

Following Relativity Principle, one thus should develop gravitation theory as a gauge theory on a principal frame bundle  $LX$  over an oriented four-dimensional smooth manifold  $X$ , called the world manifold  $X$  [4, 18].

Equivalence Principle reformulated in geometric terms requires that the structure group

$$GL_4 = GL^+(4, \mathbb{R}) \quad (1.1)$$

of a frame bundle  $LX$  and associated bundles is reducible to a Lorentz group  $SO(1, 3)$ . It means that these fibre bundles admit atlases with  $SO(1, 3)$ -valued transition functions and, equivalently, that there exist principal subbundles of  $LX$  with a Lorentz structure group (Section 5.2). This is just a case of spontaneous symmetry breaking. Accordingly, there is one-to-one correspondence between the Lorentz principal subbundles of a frame bundle  $LX$  (called the Lorentz reduced structures) and the global sections of the quotient bundle  $LX/SO(1, 3) \rightarrow X$  (5.39) which are pseudo-Riemannian metrics on a world manifold  $X$  [6, 11, 18, 21].

An underlying physical reason for Equivalence Principle is the existence of Dirac spinor fields which possess Lorentz spin symmetries, but do not admit general covariant transformations [11, 13, 18].

In classical field theory, Dirac spinor fields usually are represented by sections of a spinor bundle on a world manifold  $X$  whose typical fibre is a Dirac spinor space  $\Psi_D$  and whose structure group is a Lorentz spin group  $\text{Spin}(1, 3)$ .

Note that spinor representations of Lie algebras  $so(m, n - m)$  of pseudo-orthogonal Lie groups  $SO(m, n - m)$ ,  $n \geq 1$ ,  $m = 0, 1, \dots, n$ , were discovered by E. Cartan in 1913, when he classified finite-dimensional representations of simple Lie algebras [2]. Though, there is a problem of spinor representations of pseudo-orthogonal Lie groups  $SO(m, n - m)$  themselves. Spinor representations are attributes of Spin groups  $\text{Spin}(m, n - m)$ . Spin groups  $\text{Spin}(m, n - m)$  are two-fold coverings (3.20) of pseudo-orthogonal groups  $SO(m, n - m)$ .

Spin groups  $\text{Spin}(m, n - m)$  are defined as certain subgroups of real Clifford algebras  $\mathcal{Cl}(m, n - m)$  (3.18). Moreover, spinor representations of Spin groups in fact are the restriction of spinor representation of Clifford algebras to its Spin subgroups. Indeed, one needs an action of a whole Clifford algebra in a spinor space in order to construct a Dirac operator. In 1935, R. Brauer and H. Weyl described spinor representations in terms of Clifford algebras [1, 9].

Our approach to describing spinors is the following.

- We are based on the fact that real Clifford algebras  $\mathcal{Cl}(m, n - m)$  and complex Clifford algebras  $\mathbb{C}\mathcal{Cl}(n)$  of even dimension  $n$  are isomorphic to matrix algebras (Theorems 2.5 and 2.9, respectively). Therefore, they are simple (Corollaries 2.6 and 2.10), and all their automorphisms are inner (Theorems

3.1 and 3.7). Their invertible elements constitute general linear matrix groups (Theorems 3.2 and 3.8). They act on Clifford algebras by a left-regular representation, and their adjoint representation as projective linear groups exhaust all automorphisms of Clifford algebras (Theorems 3.3 and 3.9).

Note that this is just the case of a Clifford algebra  $\mathcal{Cl}(1, 3)$  in gravitation theory (Part II).

- Real and complex Clifford algebras of odd dimension  $n$  are described as even subrings of Clifford algebras of even dimension (Lemmas 2.4 and 2.11, Example 2.8).

- Given a real Clifford algebra  $\mathcal{Cl}(m, n - m)$ , the corresponding spinor space  $\Psi(m, n - m)$  is defined as a carrier space of its exact irreducible representation (Definition 4.1). We are based on the fact that an exact irreducible representation of a real Clifford algebra  $\mathcal{Cl}(m, n - m)$  of even dimension  $n$  is unique up to an equivalence, whereas a Clifford algebra  $\mathcal{Cl}(m, n - m)$  of odd dimension  $n$  admits two inequivalent irreducible representations (Theorem 2.7).

In particular, a Dirac spinor space is defined to be a spinor space  $\Psi(1, 3)$  of a Clifford algebra  $\mathcal{Cl}(1, 3)$  (Example 2.6).

However, Examples 2.4 – 2.5 of Clifford algebras  $\mathcal{Cl}(0, 2)$  and  $\mathcal{Cl}(2, 0)$ , respectively, show that spinor spaces  $\Psi(m, n - m)$  and  $\Psi(m', n - m')$  need not be isomorphic vector spaces for  $m' \neq m$ . For instance, a Dirac spinor space  $\Psi(1, 3)$  differs from a Majorana spinor space  $\Psi(3, 1)$  of a Clifford algebra  $\mathcal{Cl}(3, 1)$  (Example 2.7). In contrast with the four-dimensional real matrix representation (2.26) of  $\mathcal{Cl}(3, 1)$ , a representation of a Clifford algebra  $\mathcal{Cl}(3, 1)$  by complex Dirac's matrices (2.22) is not a representation of a real Clifford algebra by virtue of Definition 2.3. Indeed, from the physical viewpoint, Dirac spinor fields describing charged fermions are complex fields.

- We therefore focus our consideration on complex Clifford algebras and complex spinors. A complex Clifford algebra  $\mathbb{C}\mathcal{Cl}(n)$  (Definition 2.4) of even dimension  $n$  is isomorphic to a ring  $\text{Mat}(2^{n/2}, \mathbb{C})$  of complex  $(2^{n/2} \times 2^{n/2})$ -matrices (Theorem 2.9). Its invertible elements constitute a general linear matrix group  $GL(2^{n/2}, \mathbb{C})$  whose adjoint representation in  $\mathbb{C}\mathcal{Cl}(n)$  yields the projective linear group  $PGL(2^{n/2}, \mathbb{C})$  (3.31) of automorphisms of  $\mathbb{C}\mathcal{Cl}(n)$  (Theorem 3.7).

- Given a complex Clifford algebra  $\mathbb{C}\mathcal{Cl}(n)$ , the corresponding complex spinor space  $\Psi(n)$  is defined as a carrier space of its exact irreducible representation (Definition 4.2). Similarly to a case of real Clifford algebras, we are based on the fact that an exact irreducible representation of a complex Clifford algebra  $\mathbb{C}\mathcal{Cl}(n)$  of even dimension  $n$  is unique up to an equivalence, whereas a Clifford algebra  $\mathbb{C}\mathcal{Cl}(n)$  of odd dimension  $n$  admits two inequivalent irreducible representations (Theorem 2.12). Due to the canonical monomorphism  $\mathcal{Cl}(m, n - m) \rightarrow \mathbb{C}\mathcal{Cl}(n)$  (2.38) of real Clifford algebras to the complex ones, a complex spinor space  $\Psi(n)$  admits a representation of a real Clifford algebra  $\mathcal{Cl}(m, n - m)$ , though it need not be irreducible.

- In accordance with Definition 4.2 and Theorem 2.12, we define a particular complex Clifford space  $\Psi(n)$  in a case of even  $n$  as a minimal left ideal of a

complex Clifford algebra  $\mathbb{C}\mathcal{C}\ell(n)$  (Definition 4.3). Thus, a spinor representation

$$\gamma : \mathbb{C}\mathcal{C}\ell(n) \times \Psi(n) \rightarrow \Psi(n) \quad (1.2)$$

of a Clifford algebra  $\mathbb{C}\mathcal{C}\ell(n)$  is equivalent to the canonical representation of  $\text{Mat}(2^{n/2}, \mathbb{C})$  by matrices in a complex vector space  $\Psi(n) = \mathbb{C}^{2^{n/2}}$  (Theorem 4.1). Moreover, this spinor space  $\Psi(n)$  also carries out a left-regular irreducible representation of a general linear matrix group  $GL(2^{n/2}, \mathbb{C}) = \mathcal{G}\mathbb{C}\mathcal{C}\ell(n)$  which is equivalent to the natural matrix representation of  $GL(2^{n/2}, \mathbb{C})$  in  $\mathbb{C}^{2^{n/2}}$  (Corollary 4.2). Thus, this group preserves spinor spaces.

- We show that any complex spinor space  $\Psi(n)$  as a minimal left ideal is generated by some Hermitian idempotent  $p \in \Psi(n)$  (4.1) (Theorem 4.6), and obtain its group of automorphisms. A key point, that a spinor subspace  $\Psi(n)$  of a complex Clifford algebra  $\mathbb{C}\mathcal{C}\ell(n)$  is not unique, and it is not stable under automorphisms of  $\mathbb{C}\mathcal{C}\ell(n)$ .

Treating a complex spinor space  $\Psi(n)$  as a subspace (i.e. a minimal left ideal) of a complex Clifford algebra  $\mathbb{C}\mathcal{C}\ell(n)$  which carries out its left-regular representation (1.2), we believe reasonable to consider a fibre bundle in spinor spaces  $\Psi(n)$  as a subbundle of a fibre bundle in Clifford algebras. However, one usually considers fibre bundles in Clifford algebras whose structure group is a group of automorphisms of these algebras [4, 9] (Remark 6.1). A problem is that, as was mentioned above, this group fails to preserve spinor subspaces  $\Psi(n)$  of a Clifford algebra  $\mathbb{C}\mathcal{C}\ell(n)$  and, thus, it can not be a structure group of spinor bundles.

Therefore, we define fibre bundles  $\mathcal{C}$  (6.1) in Clifford algebras  $\mathbb{C}\mathcal{C}\ell(n)$  whose structure group is a general linear group  $GL(2^{n/2}, \mathbb{C})$  of invertible elements of  $\mathbb{C}\mathcal{C}\ell(n)$  which acts on this algebra by left multiplications (Definition 6.1). Certainly, it preserves minimal left ideals of this algebra and, consequently, is a structure group of spinor subbundles  $S$  of a Clifford algebra bundle  $\mathcal{C}$  (Definition 6.2).

In particular, let  $X$  be a smooth real manifold of dimension  $2^{n/2}$ ,  $n = 2, 4, \dots$ . Let  $TX$  be the tangent bundle over  $X$ . Their structure group is  $GL(2^{n/2}, \mathbb{R})$ . Due to the canonical group monomorphism  $GL(2^{n/2}, \mathbb{R}) \rightarrow GL(2^{n/2}, \mathbb{C})$  6.6, the complexification  $\mathbb{C}TX$  (6.7) of  $TX$  can be represented as a spinor bundle (Remark 6.2). This bundle admits general covariant transformations and, thus, it is a natural bundle.

It should be emphasized that, though there is the ring monomorphism  $\mathcal{C}\ell(m, n-m) \rightarrow \mathbb{C}\mathcal{C}\ell(n)$  (2.38), the Clifford algebra bundle  $\mathcal{C}$  (6.1) need not contain a subbundle in real Clifford algebras  $\mathcal{C}\ell(m, n-m)$  unless a structure group  $GL(2^{n/2}, \mathbb{C})$  of  $\mathcal{C}$  is reducible to a group  $\mathcal{G}\mathcal{C}\ell(m, n-m)$  of invertible elements of  $\mathcal{C}\ell(m, n-m)$ . We study this condition (Section 6.1). Let  $X$  be an  $n$ -dimensional smooth manifold and  $LX$  a principal frame bundle over  $X$ . We show that any global section  $h$  of the quotient bundle  $\Sigma(m, n-m) = LX/O(m, n-m) \rightarrow X$  (6.13) is associated to the fibre bundle  $\mathcal{C}^h \rightarrow X$  (6.11) in complex Clifford algebras  $\mathbb{C}\mathcal{C}\ell(n)$  which contains the subbundle  $\mathcal{C}^h(m, n-m) \rightarrow X$  (6.12) in real Clifford algebras  $\mathcal{C}\ell(m, n-m)$  and a spinor subbundle  $S^h \rightarrow X$ .

A key point is that, given different sections  $h$  and  $h'$  of the quotient bundle  $\Sigma(m, n-m) \rightarrow X$  (6.13), the Clifford algebra bundles  $\mathcal{C}^h$  and  $\mathcal{C}^{h'}$  need not be isomorphic. In order to describe all these non-isomorphic Clifford algebra bundles  $\mathcal{C}^h$ , follow a construction of composite bundles (Section 6.2). We consider composite Clifford algebra bundles  $\mathcal{C}_\Sigma$  (6.18) and  $\mathcal{C}(m, n-m)_\Sigma$  (6.19), and the spinor bundle  $S_\Sigma$  (6.20) over a base  $\Sigma(m, n-m)$  (6.13). Then given a global section  $h$  of the quotient bundle  $\Sigma(m, n-m) \rightarrow X$  (6.13), the pull-back bundles  $h^*\mathcal{C}_\Sigma$ ,  $h^*\mathcal{C}(m, n-m)_\Sigma$  and  $h^*S_\Sigma$  are the above mentioned fibre bundles  $\mathcal{C}^h \rightarrow X$ ,  $\mathcal{C}^h(m, n-m) \rightarrow X$  and  $S^h \rightarrow X$ , respectively.

This is just the case of gravitation theory where, in order to define a Dirac operator, we must consider a fibre bundle in Clifford algebras  $\mathcal{Cl}(1, 3)$  whose generating spaces are cotangent spaces to a world manifold  $X$ .

In forthcoming Part II of our work, following the technique of composite Clifford algebra bundles in Section 6.2, we consider composite Clifford algebra bundles  $\mathcal{C}_\Sigma$  and  $\mathcal{C}(1, 3)_\Sigma$ , and a spinor bundle  $S_\Sigma$  over the base  $LX/SO(1, 3)$  (5.39). As was mentioned above, global sections  $h$  of the quotient bundle  $LX/SO(1, 3) \rightarrow X$  are pseudo-Riemannian metrics on  $X$ . Given such a section, the corresponding pull-back bundles  $h^*\mathcal{C}_\Sigma$ ,  $h^*\mathcal{C}(1, 3)_\Sigma$  and  $h^*S_\Sigma$  are  $h$ -associated fibre bundles  $\mathcal{C}^h \rightarrow X$ ,  $\mathcal{C}^h(1, 3) \rightarrow X$  and  $S^h \rightarrow X$  over  $X$ , respectively.

## 2 Clifford algebras

A real Clifford algebra is defined as a ring (i.e., a unital associative algebra) possessing a certain vector subspace of generating elements (Definition 2.1). However, such a ring can possess different generating spaces. Therefore, we also consider a real Clifford algebra without specifying its generating space. Complex Clifford algebras are defined as the complexification of the real ones (Definition 2.4).

### 2.1 Real Clifford algebras

Let  $V = \mathbb{R}^n$  be an  $n$ -dimensional real vector space provided with a non-degenerate bilinear form (a pseudo-Euclidean metric)  $\eta$ . Let us consider a tensor algebra

$$\otimes V = \mathbb{R} \oplus V \oplus \otimes^2 V \oplus \dots \oplus \otimes^k V \oplus \dots$$

of  $V$  and its two-sided ideal  $I_\eta$  generated by the elements

$$v \otimes v' + v' \otimes v - 2\eta(v, v')e, \quad v, v' \in V,$$

where  $e$  denotes the unit element of  $\otimes V$ . The quotient  $\otimes V/I_\eta$  is a real non-commutative ring.

**DEFINITION 2.1:** A real ring  $\otimes V/I_\eta$  together with a fixed generating subspace  $(V, \eta)$  is called the real Clifford algebra  $\mathcal{Cl}(V, \eta)$  modelled over a pseudo-Euclidean space  $(V, \eta)$ .  $\square$

**Remark 2.1:** Unless otherwise stated, by a Clifford algebra hereafter is meant a real Clifford algebra in Definition 2.1.  $\square$

There is the canonical monomorphism of a real vector space  $V$  to the quotient  $\otimes V/I_\eta$ . It is a generating subspace of a real ring  $\otimes V/I_\eta$ . Its elements obey the relations

$$vv' + v'v - 2\eta(v, v')e = 0, \quad v, v' \in V.$$

DEFINITION 2.2: Given Clifford algebras  $\mathcal{C}\ell(V, \eta)$  and  $\mathcal{C}\ell(V', \eta')$ , by their isomorphism is meant an isomorphism of them as rings:

$$\phi : \mathcal{C}\ell(V, \eta) \rightarrow \mathcal{C}\ell(V', \eta'), \quad \phi(qq') = \phi(q)\phi(q'), \quad (2.1)$$

which also is an isometric isomorphism of their generating pseudo-Euclidean spaces:

$$\begin{aligned} \phi : \mathcal{C}\ell(V, \eta) \supset (V, \eta) &\rightarrow (V', \eta') \subset \mathcal{C}\ell(V', \eta'), \\ 2\eta'(\phi(v), \phi(v')) &= \phi(v)\phi(v') + \phi(v')\phi(v) = \phi(vv' + v'v) = 2\eta(v, v'). \end{aligned} \quad (2.2)$$

$\square$

It follows from the isomorphism (2.2) that two Clifford algebras  $\mathcal{C}\ell(V, \eta)$  and  $\mathcal{C}\ell(V', \eta')$  are isomorphic iff they are modelled over pseudo-Euclidean spaces  $(V, \eta)$  and  $(V', \eta')$  of the same signature. Let a pseudo-Euclidean metric  $\eta$  be of signature  $(m; n - m) = (1, \dots, 1; -1, \dots, -1)$ . Let  $\{v^1, \dots, v^n\}$  be a basis for  $V$  such that  $\eta$  takes a diagonal form

$$\eta^{ab} = \eta(v^a, v^b) = \pm \delta^{ab}.$$

Then a ring  $\mathcal{C}\ell(V, \eta)$  is generated by elements  $v^1, \dots, v^n$  which obey relations

$$v^a v^b + v^b v^a = 2\eta^{ab}e.$$

We agree to call  $\{v^1, \dots, v^n\}$  the basis for a Clifford algebra  $\mathcal{C}\ell(\mathbb{R}^n, \eta)$ . Given this basis, let us denote  $\mathcal{C}\ell(\mathbb{R}^n, \eta) = \mathcal{C}\ell(m, n - m)$ .

In accordance with Definition 2.2, any isomorphism (2.1) – (2.2) of Clifford algebras is their ring isomorphism (2.1). However, the converse is not true, because their ring isomorphism (2.1) need not be the isometric isomorphism (2.2) of their generating spaces. Therefore, we also consider Clifford algebras, without specifying their generating spaces.

LEMMA 2.1: Any isometric isomorphism (2.2) of generating vector spaces  $\phi : V \rightarrow V'$  of Clifford algebras  $\mathcal{C}\ell(V, \eta)$  and  $\mathcal{C}\ell(V', \eta')$  is prolonged to their ring isomorphism (2.1):

$$\phi : \mathcal{C}\ell(V, \eta) \rightarrow \mathcal{C}\ell(V', \eta') \quad \phi(v_1 \cdots v_k) = \phi(v_1) \cdots \phi(v_k), \quad (2.3)$$

which also is an isomorphism of Clifford algebras.  $\square$

**Remark 2.2:** Let  $g$  be a general (non-isometric) linear automorphism of a generating vector space  $V$  of a Clifford algebra  $\mathcal{Cl}(V, \eta)$ . It yields an automorphism of  $\mathcal{Cl}(V, \eta)$  as a real vector space, but not its ring automorphism because

$$\begin{aligned} g(v)g(v') + g(v')g(v) &= 2\eta(g(v), g(v'))e \neq \\ 2\eta(v, v')e &= g(vv' + v'v), \quad v, v' \in V, \end{aligned}$$

in general. Let us provide a vector space  $V$  with a different pseudo-Euclidean metric  $\eta'$  such that

$$\eta'(g(v), g(v')) = \eta(v, v'), \quad v, v' \in V.$$

It is of the same signature as  $\eta$ . Then a morphism  $g$  is an isometric isomorphism of a pseudo-Euclidean space  $(V, \eta)$  to a pseudo-Euclidean space  $(V, \eta')$ . Accordingly it yields an isomorphism of a Clifford algebra  $\mathcal{Cl}(V, \eta)$  to a Clifford algebra  $\mathcal{Cl}(V, \eta')$  modelled over  $(V, \eta')$ .  $\square$

**Example 2.3:** There are the following isomorphisms of real rings [9]:

$$\mathcal{Cl}(1, 0) = \mathbb{R} \oplus \mathbb{R}, \quad (2.4)$$

$$\mathcal{Cl}(0, 1) = \mathbb{C}, \quad (2.5)$$

Let  $\{r^1 = 1, r^2 = 1\}$  be a basis for a ring  $\mathbb{R} \oplus \mathbb{R}$ . Then the isomorphism (2.4) reads ( $e \leftrightarrow r^1$ ),  $v^1 \leftrightarrow r^2$ . Accordingly, the isomorphism (2.5) takes a form  $e \leftrightarrow 1$ ,  $v^1 \leftrightarrow i$ .  $\square$

**Example 2.4:** There is a ring isomorphism

$$\mathcal{Cl}(0, 2) = \mathbb{H}, \quad (2.6)$$

where  $\mathbb{H}$  is a real division ring of quaternions. An underlying real vector space of  $\mathbb{H}$  has a basis  $\{\mathbf{1}, \tau^1, \tau^2, \tau^3\}$  whose elements obey the relations

$$(\tau^1)^2 = (\tau^2)^2 = (\tau^3)^2 = \tau^1\tau^2\tau^3 = -\mathbf{1},$$

where  $\mathbf{1}$  is the unit element of  $\mathbb{H}$ . These relations define the real division ring  $\mathbb{H}$  with two generating elements, e.g.,  $\tau^1$  and  $\tau^2$ . We have

$$\tau^1\tau^2 = -\tau^2\tau^1 = \tau^3, \quad \tau^2\tau^3 = -\tau^3\tau^2 = \tau^1, \quad \tau^3\tau^1 = -\tau^1\tau^3 = \tau^2.$$

Due to an isomorphism

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} = \text{Mat}(2, \mathbb{C}), \quad (2.7)$$

a quaternion division ring  $\mathbb{H}$  can be represented as a real subalgebra of an algebra  $\text{Mat}(2, \mathbb{C})$  of complex  $(2 \times 2)$ -matrices whose underlying real vector space possesses a basis

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau^1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \tau^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \tau^3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad (2.8)$$



so that  $\tau^k = -i\sigma^k$ ,  $k = 1, 2, 3$ , where  $\sigma^k$  are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.9)$$

Then the isomorphism  $\mathcal{C}\ell(0, 2) = \mathbb{H}$  (2.6) can be written in a form  $v^1 \leftrightarrow \tau^1$ ,  $v^2 \leftrightarrow \tau^2$ , but it is not canonical. Using this isomorphism and the matrix representation (2.8) of a quaternion division ring  $\mathbb{H}$ , we obtain a matrix representation of a Clifford algebra  $\mathcal{C}\ell(0, 2)$  as a real subalgebra of an algebra  $\text{Mat}(2, \mathbb{C})$ . Its underlying real vector space possesses a basis

$$e = \mathbf{1}, \quad v^k = \tau^k, \quad k = 1, 2, 3. \quad (2.10)$$

□

It may happen that a ring  $\mathcal{C}\ell(V, \eta)$  admits a generating pseudo-Euclidean space  $(V', \eta')$  whose signature differs from that of  $(V, \eta)$ . In this case,  $\mathcal{C}\ell(V, \eta)$  possesses the structure of a Clifford algebra  $\mathcal{C}\ell(V', \eta')$  which is not isomorphic to a Clifford algebra  $\mathcal{C}\ell(V, \eta)$ .

LEMMA 2.2: There are ring isomorphisms

$$\mathcal{C}\ell(m, n - m) = \mathcal{C}\ell(n - m + 1, m - 1), \quad (2.11)$$

$$\mathcal{C}\ell(m, n - m) = \mathcal{C}\ell(m - 4, n - m + 4), \quad n, m \geq 4. \quad (2.12)$$

□

*Proof:* Let us consider a Clifford algebra  $\mathcal{C}\ell(m, n - m)$  of  $m > 0, n > 1$ , possessing a basis  $\{v^1, \dots, v^n\}$ . A ring  $\mathcal{C}\ell(m, n - m)$  also is generated by elements

$$w^1 = v^1, \quad w^i = v^1 v^i, \quad i > 1. \quad (2.13)$$

These elements obey the relations

$$\begin{aligned} w^i w^k + w^k w^i &= 2\eta^{ik} e, \\ \eta'^{11} &= \eta^{11}, \quad \eta'^{1k} = 0, \quad \eta'^{ik} = -\eta^{11} \eta^{ik}, \quad i, k > 1. \end{aligned}$$

Hence, a ring  $\mathcal{C}\ell(m, n - m)$  also is a Clifford algebra modelled over a pseudo-Euclidean space  $(\mathbb{R}^n, \eta')$  of signature  $(1 + n - m; m - 1)$ . Thus, we have the ring isomorphism (2.11) given by the relations (2.13). Turn now to the isomorphism (2.12). Let  $(v^0, v^1, v^2, v^3, v^i)$  and  $(w^0, w^1, w^2, w^3, w^i)$  be bases for Clifford algebras  $\mathcal{C}\ell(m, n - m)$  and  $\mathcal{C}\ell(m - 4, n - m + 4)$ , respectively. Then their isomorphism (2.12) is given by identifications  $w^i \leftrightarrow v^i$  and

$$w^0 \leftrightarrow v^1 v^2 v^3, \quad w^1 \leftrightarrow v^0 v^2 v^3, \quad w^2 \leftrightarrow v^0 v^1 v^3, \quad w^3 \leftrightarrow v^0 v^1 v^2. \quad (2.14)$$

□

**Example 2.5:** We have a ring isomorphism

$$\mathcal{C}\ell(2, 0) = \mathcal{C}\ell(1, 1) = \text{Mat}(2, \mathbb{R}), \quad (2.15)$$

where  $\text{Mat}(2, \mathbb{R})$  is a real ring of  $(2 \times 2)$ -matrices. It exemplifies the isomorphism (2.11) which reads:  $w^1 \leftrightarrow e^1$ ,  $w^2 \rightarrow e^1 e^2$ , where  $\{e^1, e^2\}$  and  $\{w^1, w^2\}$  are the bases for  $\mathcal{C}\ell(2, 0)$  and  $\mathcal{C}\ell(1, 1)$ , respectively. The matrix representation (2.15) of  $\mathcal{C}\ell(2, 0)$  by  $\text{Mat}(2, \mathbb{R})$  takes a form

$$e^1 = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e^2 = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.16)$$

Accordingly, the matrix representation  $\mathcal{C}\ell(1, 1)$  by  $\text{Mat}(2, \mathbb{R})$  is

$$w^1 = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad w^2 = \tau^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (2.17)$$

The real rings  $\mathcal{C}\ell(2, 0)$  (2.16) and  $\mathcal{C}\ell(1, 1)$  (2.17) coincide with each other. Their underlying vector space in  $\text{Mat}(2, \mathbb{R})$  possesses a basis  $\{\mathbf{1}, \sigma^1, \sigma^3, \tau^2\}$ .  $\square$

With the real ring isomorphism (2.15), we obtain the following recursion relation.

LEMMA 2.3: There is a real ring isomorphism

$$\mathcal{C}\ell(p+1, q+1) = \mathcal{C}\ell(1, 1) \otimes \mathcal{C}\ell(p, q) = \text{Mat}(2, \mathcal{C}\ell(p, q)), \quad (2.18)$$

where  $\text{Mat}(2, \mathcal{C}\ell(p, q))$  is an algebra of  $2 \times 2$  matrices with entries in  $\mathcal{C}\ell(p, q)$ .  $\square$

*Proof:* The isomorphisms (2.18) take a form

$$\begin{aligned} v^+ &= w^1 \otimes e = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, & v^- &= w^2 \otimes e = \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \\ v^i &= w^1 w^2 \otimes e^i = \begin{pmatrix} \tau^i & 0 \\ 0 & -\tau^i \end{pmatrix}, \end{aligned} \quad (2.19)$$

where  $\{v^i, v^+, v^-\}$  is a basis for  $\mathcal{C}\ell(p+1, q+1)$ ,  $\{e^i\}$  is that for  $\mathcal{C}\ell(p, q)$  and  $\{w^1, w^2\}$  is the basis (2.17) for  $\mathcal{C}\ell(1, 1)$ .  $\square$

**Example 2.6:** Using isomorphisms (2.6), (2.11), (2.12), (2.15) and (2.18), one can obtain the real ring isomorphisms

$$\mathcal{C}\ell(1, 3) = \mathcal{C}\ell(4, 0) = \mathcal{C}\ell(0, 4) = \mathcal{C}\ell(1, 1) \otimes \mathcal{C}\ell(0, 2) = \text{Mat}(2, \mathbb{H}), \quad (2.20)$$

The isomorphism  $\mathcal{C}\ell(4, 0) = \mathcal{C}\ell(0, 4)$  (2.20) exemplifies the isomorphism (2.12) given by the identification (2.14). In view of the formulas (2.10) and (2.19), the matrix representation  $\mathcal{C}\ell(1, 3) = \text{Mat}(2, \mathbb{H})$  (2.20) reads

$$v^+ = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad v^- = \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad v^{1,2} = \begin{pmatrix} -i\sigma^{1,2} & 0 \\ 0 & i\sigma^{1,2} \end{pmatrix}, \quad (2.21)$$

where  $\sigma^{1,2}$  are the Pauli matrices (2.9). Let us call it the standard representation, though it is not canonical. In particular, one usually deal with a representation of  $\mathcal{C}\ell(1, 3)$  by Dirac's matrices

$$v^0 = \gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \quad v^j = \gamma^j = \begin{pmatrix} 0 & -\sigma^j \\ \sigma^j & 0 \end{pmatrix}. \quad (2.22)$$

Let us also mention its different representation by other Dirac's matrices

$$\begin{aligned} \tilde{\gamma}^\mu &= S\gamma^\mu S^{-1}, \quad S = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1} & -\mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}, \\ \tilde{\gamma}^0 &= \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad \tilde{\gamma}^j = \begin{pmatrix} 0 & -\sigma^j \\ \sigma^j & 0 \end{pmatrix}. \end{aligned} \quad (2.23)$$

The isomorphism  $\mathcal{C}\ell(4, 0) = \mathcal{C}\ell(1, 3)$  (2.20) exemplifies the isomorphisms (2.11). Given the matrix representation (2.23) of  $\mathcal{C}\ell(1, 3)$ , it provides the matrix representation

$$w^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad w^j = \begin{pmatrix} \sigma^j & 0 \\ 0 & -\sigma^j \end{pmatrix} \quad (2.24)$$

of a Clifford algebra  $\mathcal{C}\ell(4, 0)$   $\square$

**Example 2.7:** Using isomorphisms (2.11), (2.15) and (2.18), one can obtain the real ring isomorphisms

$$\mathcal{C}\ell(2, 2) = \mathcal{C}\ell(3, 1) = \mathcal{C}\ell(1, 1) \otimes \mathcal{C}\ell(0, 2) = \text{Mat}(4, \mathbb{R}). \quad (2.25)$$

The formulas (2.16) and (2.18) lead to the representation (2.25) of  $\mathcal{C}\ell(3, 1)$  by real matrices:

$$\begin{aligned} v^+ &= \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad v^- = \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \\ v^1 &= \begin{pmatrix} \sigma^1 & 0 \\ 0 & -\sigma^1 \end{pmatrix}, \quad v^2 = \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}, \end{aligned} \quad (2.26)$$

It is an irreducible four-dimensional representation of a Clifford algebra  $\mathcal{C}\ell(3, 1)$ . By virtue of Theorem 2.7, this irreducible representation is unique up to an equivalence.  $\square$

Let  $\mathcal{C}\ell^0(m, n - m)$  be a vector subspace of elements of a Clifford algebra  $\mathcal{C}\ell(m, n - m)$  which is spanned by polynomials in elements of  $\mathbb{R}^n$  of even degree. It is obviously a subring of a ring  $\mathcal{C}\ell(m, n - m)$ , called its even subring.

LEMMA 2.4: There exists a ring isomorphism

$$\mathcal{C}\ell^0(m, n - m) = \mathcal{C}\ell(n - m, m - 1), \quad n > 1. \quad (2.27)$$

$\square$

*Proof:* Let  $\{v^0, \dots, v^{n-1}\}$  and  $\{w^1, \dots, w^{n-1}\}$  be bases for  $\mathcal{C}\ell(m, n-m)$  and  $\mathcal{C}\ell(m, n-m-1)$ . Then the isomorphism (2.27) is defined by the identification  $w^i = v^0 v^i$ .  $\square$

**Example 2.8:** Let us consider a Clifford algebra

$$\mathcal{C}\ell(0, 3) = \mathcal{C}\ell^0(4, 0). \quad (2.28)$$

Let a Clifford algebra  $\mathcal{C}\ell(4, 0)$  be represented by the matrices  $(\tilde{\gamma}^0, -i\tilde{\gamma}^j)$  (2.23). Then a Clifford algebra  $\mathcal{C}\ell(0, 3)$  is generated by matrices

$$\begin{aligned} (a_0 \tilde{\gamma}^0 + ia_i \tilde{\gamma}^i)(b_0 \tilde{\gamma}^0 + ib_i \tilde{\gamma}^i) &= (a_0 \mathbf{1} + ia_i \tilde{\gamma}^i \tilde{\gamma}^0)(b_0 \mathbf{1} + ib_i \tilde{\gamma}^0 \tilde{\gamma}^i) = \\ &= \begin{pmatrix} c_0 \mathbf{1} + c_i \tau^i & 0 \\ 0 & d_0 \mathbf{1} + d_i \tau^i \end{pmatrix}, \quad c_\mu, d_\mu \in \mathbb{R}. \end{aligned} \quad (2.29)$$

Thus, there is a real ring isomorphism

$$\mathcal{C}\ell(0, 3) = \mathbb{H} \times \mathbb{H}. \quad (2.30)$$

$\square$

The recursion relation (2.3) and the ring isomorphisms (2.4), (2.5), (2.6), (2.15) and (2.30) enable us to provide the matrix representation of any real Clifford algebra as follows.

**THEOREM 2.5:** Clifford algebras  $\mathcal{C}\ell(p, q)$  as rings are isomorphic to the following matrix algebras.

$$\mathcal{C}\ell(p, q) = \begin{cases} \text{Mat}(2^{(p+q)/2}, \mathbb{R}) = \bigotimes_{\mathbb{R}}^{(p+q)/2} \text{Mat}(2, \mathbb{R}) & p - q = 0, 2 \pmod{8} \\ \text{Mat}(2^{(p+q-1)/2}, \mathbb{R}) \oplus \text{Mat}(2^{(p+q-1)/2}, \mathbb{R}) & p - q = 1 \pmod{8} \\ \text{Mat}(2^{(p+q-1)/2}, \mathbb{C}) & p - q = 3, 7 \pmod{8} \\ \text{Mat}(2^{(p+q-2)/2}, \mathbb{H}) & p - q = 4, 6 \pmod{8} \\ \text{Mat}(2^{(p+q-3)/2}, \mathbb{H}) \oplus \text{Mat}(2^{(p+q-3)/2}, \mathbb{H}) & p - q = 5 \pmod{8} \end{cases} \quad (2.31)$$

$\square$

*Proof:* Owing to the isomorphism (2.12), a Clifford algebra  $\mathcal{C}\ell(p, q)$  is isomorphic to a Clifford algebra  $\mathcal{C}\ell(p-4k, q+4k)$ ,  $k \in \mathbb{Z}$ , so that  $p-q-8k < 8$ . The we have eight different algebras

$$\begin{aligned} \mathcal{C}\ell((p+q)/2, (p+q)/2) & \quad p - q = 0, 2 \pmod{8} \\ \mathcal{C}\ell((p+q+1)/2, (p+q-1)/2) & \quad p - q = 1 \pmod{8} \\ \mathcal{C}\ell((p+q+3)/2, (p+q-3)/2) & \quad p - q = 3, 7 \pmod{8} \\ \mathcal{C}\ell((p+q-2)/2, (p+q+2)/2) & \quad p - q = 4, 6 \pmod{8} \\ \mathcal{C}\ell((p+q-3)/2, (p+q+3)/2) & \quad p - q = 5 \pmod{8}. \end{aligned}$$

Then the relations (2.3) leads to the isomorphisms

$$\begin{aligned}
\text{Mat}(2^{(p+q)/2}, \mathbb{R}) &= \bigotimes_{\mathbb{R}}^{(p+q)/2} \text{Mat}(2, \mathbb{R}) & p - q = 0, 2 \pmod{8} \\
\text{Mat}(2^{(p+q-1)/2}, \mathcal{C}\ell(1, 0)) & & p - q = 1 \pmod{8} \\
\text{Mat}(2^{(p+q-1)/2}, \mathcal{C}\ell(0, 1)) & & p - q = 3, 7 \pmod{8} \\
\text{Mat}(2^{(p+q-2)/2}, \mathcal{C}\ell(0, 2)) & & p - q = 4, 6 \pmod{8} \\
\text{Mat}(2^{(p+q-3)/2}, \mathcal{C}\ell(0, 3)) & & p - q = 5 \pmod{8}
\end{aligned}$$

The result (2.31) follows from the isomorphisms (2.4), (2.5), (2.6), (2.15) and (2.30).  $\square$

**COROLLARY 2.6:** Since matrix algebras  $\text{Mat}(r, \mathcal{K})$ ,  $\mathcal{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ , are simple, a glance at Table 2.31 shows that real Clifford algebras  $\mathcal{C}\ell(V, \eta)$  modelled over even dimensional vector spaces  $V$  (i.e.,  $p - q$  is even) are simple.  $\square$

**DEFINITION 2.3:** By a representation of a Clifford algebra  $\mathcal{C}\ell(V, \eta)$  is meant its ring homomorphism  $\rho$  to a real ring of linear endomorphisms of a finite-dimensional real vector space  $\Xi$ , whose dimension is called the dimension of a representation.  $\square$

For instance, the real matrix representation (2.26) of a real Clifford algebra  $\mathcal{C}\ell(3, 1)$  is its representation in accordance with Definition 2.3. At the same time, a representation of a Clifford algebra  $\mathcal{C}\ell(3, 1)$  by Dirac's matrices (2.22) is not that by Definition 2.3.

A representation is said to be exact if  $\rho$  is an isomorphism. A representation is called irreducible if there is no proper subspace of  $\Xi$  which is a carrier space of a representation of  $\mathcal{C}\ell(V, \eta)$ .

Two representations  $\rho$  and  $\rho'$  of a Clifford algebra  $\mathcal{C}\ell(V, \eta)$  in vector spaces  $\Xi$  and  $\Xi'$  are said to be equivalent if there is an isomorphism  $\xi : \Xi \rightarrow \Xi'$  of these vector spaces such that  $\rho' = \xi \circ \rho \circ \xi^{-1}$  is a real ring isomorphism of  $\rho(\mathcal{C}\ell(V, \eta))$  and  $\rho'(\mathcal{C}\ell(V, \eta))$ .

The following is a corollary of Theorem 2.5.

**THEOREM 2.7:** If  $n = \dim V$  is even, an exact irreducible representation of a real ring  $\mathcal{C}\ell(m, n - m)$  is unique up to an equivalence [9]. If  $n$  is odd there exist two inequivalent exact irreducible representations of a Clifford algebra  $\mathcal{C}\ell(m, n - m)$ .  $\square$

## 2.2 Complex Clifford algebras

Let us consider the complexification

$$\mathbb{C}\mathcal{C}\ell(m, n - m) = \mathbb{C} \otimes_{\mathbb{R}} \mathcal{C}\ell(m, n - m) \quad (2.32)$$

of a real ring  $\mathcal{C}\ell(m, n - m)$ . It is readily observed that all complexifications  $\mathbb{C}\mathcal{C}\ell(m, n - m)$ ,  $m = 0, \dots, n$ , are isomorphic:

$$\mathbb{C}\mathcal{C}\ell(m, n - m) = \mathbb{C}\mathcal{C}\ell(m', n - m'), \quad (2.33)$$

both as real and complex rings. Namely, with the bases  $\{v^i\}$  and  $\{e^i\}$  for  $\mathcal{C}\ell(m, n-m)$  and  $\mathcal{C}\ell(n, 0)$ , their isomorphisms (2.33) are given by associations

$$v^{1,\dots,m} \rightarrow e^{1,\dots,m}, \quad v^{m+1,\dots,n} \rightarrow ie^{m+1,\dots,n}. \quad (2.34)$$

Though the isomorphisms (2.34) are not unique, one can speak about an abstract complex ring  $\mathbb{C}\mathcal{C}\ell(n)$  (2.33) so that, given a real Clifford algebra  $\mathcal{C}\ell(m, n-m)$  and its complexification  $\mathbb{C}\mathcal{C}\ell(m, n-m)$  (2.32), there exist the complex ring isomorphism (2.34) of  $\mathbb{C}\mathcal{C}\ell(m, n-m)$  to  $\mathbb{C}\mathcal{C}\ell(n)$ .

**DEFINITION 2.4:** We call  $\mathbb{C}\mathcal{C}\ell(n)$  (2.33) the complex Clifford algebra, and define it as a complex ring

$$\mathbb{C}\mathcal{C}\ell(n) = \mathbb{C} \otimes_{\mathbb{R}} \mathcal{C}\ell(n, 0), \quad (2.35)$$

generated by  $n$  elements  $(e^i)$  such that

$$e^i e^j + e^j e^i = 2\kappa(e^i, e^j)e = 2\delta^{ij}e. \quad (2.36)$$

□

Let us call  $\{e^i\}$  (2.36) the Euclidean basis for a complex Clifford algebra  $\mathbb{C}\mathcal{C}\ell(n)$ . With this basis, any element of  $\mathbb{C}\mathcal{C}\ell(n)$  takes a form

$$a = \lambda e + \sum_{1 \leq k \leq n} \sum_{i_1 < \dots < i_k} \lambda_{i_1 \dots i_k} e^{i_1} \dots e^{i_k}, \quad \lambda, \lambda_{i_1 \dots i_k} \in \mathbb{C}. \quad (2.37)$$

**DEFINITION 2.5:** A complex vector space  $\mathcal{V}$ , spanned by an Euclidean basis  $\{e^i\}$  and provided with the bilinear form  $\kappa$  (2.36), is termed the Euclidean generating space of a complex Clifford algebra  $\mathbb{C}\mathcal{C}\ell(n)$ . □

**Remark 2.9:** Any generating space  $(\mathcal{V}, \kappa)$  of a complex Clifford algebra is the Euclidean one with respect to some basis of  $\mathcal{V}$ . □

**LEMMA 2.8:** The complex ring  $\mathbb{C}\mathcal{C}\ell(n)$  (2.35) possesses a canonical real subring

$$\mathcal{C}\ell(m, n-m) \rightarrow \mathbb{C}\mathcal{C}\ell(n) \quad (2.38)$$

with a basis

$$\{e^1, \dots, e^m, ie^{m+1}, \dots, ie^n\}. \quad (2.39)$$

□

**Remark 2.10:** The definition (2.35) enables us to provide a complex Clifford algebra  $\mathbb{C}\mathcal{C}\ell(n)$  with an involution

$$(\lambda e^i)^* = \bar{\lambda} e^i, \quad (e^i e^j)^* = e^j e^i, \quad \lambda \in \mathbb{C}, \quad (2.40)$$

so that an involution of the element  $a \in \mathbb{C}\mathcal{C}\ell(n)$  (2.37) reads

$$a^* = \bar{\lambda} e + \sum_{1 \leq k \leq n} \sum_{i_1 < \dots < i_k} \bar{\lambda}_{i_1 \dots i_k} e^{i_k} \dots e^{i_1}. \quad (2.41)$$

In particular, it follows that

$$a^*a = \left( \bar{\lambda}\lambda + \sum_{1 \leq k \leq n} \sum_{i_1 < \dots < i_k} \bar{\lambda}_{i_1 \dots i_k} \lambda_{i_1 \dots i_k} \right) e + \dots \neq 0. \quad (2.42)$$

An element  $a \in \mathbb{CCl}(n)$  is called the Hermitian one if  $a^* = a$ . In this case,  $a^2 \neq 0$  in accordance with the formula (2.42). The involution  $*$  (2.40) makes a complex Clifford algebra involutive. However, an automorphism of  $\mathbb{CCl}(n)$  need not be its automorphism as an involutive algebra (Remark 3.10).  $\square$

Theorem 2.5 provides the following classification of the complex Clifford algebras  $\mathbb{CCl}(n)$  (2.35).

**THEOREM 2.9:** Complex Clifford algebras are isomorphic to the following matrix ones

$$\mathbb{CCl}(n) = \begin{cases} \text{Mat}(2^{n/2}, \mathbb{C}) = \bigotimes_{\mathbb{C}}^{n/2} \text{Mat}(2, \mathbb{C}) = \bigotimes_{\mathbb{C}}^{n/2} \mathbb{CCl}(2) & n = 0 \pmod{2} \\ \text{Mat}(2^{(n-1)/2}, \mathbb{C}) \oplus \text{Mat}(2^{(n-1)/2}, \mathbb{C}) & n = 1 \pmod{2} \end{cases} \quad (2.43)$$

$\square$

**COROLLARY 2.10:** Since matrix algebras  $\text{Mat}(n, \mathbb{C})$  are simple and central (i.e., their center is proportional to the unit matrix), complex Clifford algebras  $\mathbb{CCl}(n)$  of even  $n$  are central simple algebras.  $\square$

**LEMMA 2.11:** It follows from Definition 2.4 and Lemma 2.4, that complex Clifford algebra of odd dimension are even subrings of Complex Clifford algebras of even dimension in Corollary 2.10.  $\square$

**Example 2.11:** Let us consider a complex Clifford algebra  $\mathbb{CCl}(2)$ . There is its isomorphism (2.43):

$$\mathbb{CCl}(2) = \text{Mat}(2, \mathbb{C}). \quad (2.44)$$

Its Euclidean basis in this representation is

$$e^1 = \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e^2 = \rho^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then its elements  $M$  with respect to this basis take a form

$$\text{Mat}(2, \mathbb{C}) \ni M = ae + a_1e^1 + a_2e^2 + be^1e^2, \quad a, a_1, a_2, b \in \mathbb{C},$$

so that  $M^* = M^+$  is a complex conjugate transposition of a matrix  $M \in \text{Mat}(2, \mathbb{C})$ .  $\square$

**Example 2.12:** Let us consider a complex Clifford algebra  $\mathbb{CCl}(4)$ . There is its isomorphism (2.43):

$$\mathbb{CCl}(4) = \text{Mat}(4, \mathbb{C}), \quad (2.45)$$

such that  $M^* = M^+$  is a complex conjugate transposition of a matrix  $M \in \text{Mat}(4, \mathbb{C})$ . Let  $\mathbb{C}\mathcal{C}\ell(4)$  (2.45) be generated by the elements (2.24):

$$\epsilon^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad \epsilon^j = \begin{pmatrix} \sigma^j & 0 \\ 0 & -\sigma^j \end{pmatrix} \quad (2.46)$$

which obey the relations (2.36). Let us introduce the notation

$$\begin{aligned} \epsilon^{\alpha\beta} &= \frac{1}{4}(\epsilon^\alpha \epsilon^\beta - \epsilon^\beta \epsilon^\alpha), & (\epsilon^{\alpha\beta})^2 &= -\frac{1}{4}e, \\ [\epsilon^{\alpha\beta}, \epsilon^{\mu\nu}] &= \delta^{\alpha\nu} \epsilon^{\beta\mu} + \delta^{\beta\mu} \epsilon^{\alpha\nu} - \delta^{\alpha\mu} \epsilon^{\beta\nu} - \delta^{\beta\nu} \epsilon^{\alpha\mu} \\ \epsilon^5 &= \epsilon^0 \epsilon^1 \epsilon^2 \epsilon^3, & (\epsilon^5)^2 &= e, & \epsilon^\mu \epsilon^5 &= -\epsilon^5 \epsilon^\mu, \\ \varepsilon^\mu &= \epsilon^\mu \epsilon^5, & \varepsilon^\mu \varepsilon^\nu + \varepsilon^\nu \varepsilon^\mu &= -2\delta^{\mu\nu} e. \end{aligned} \quad (2.47)$$

$$(2.48)$$

Then in accordance with the isomorphism (2.45), any element of  $M \in \text{Mat}(4, \mathbb{C})$  is represented by a sum

$$M = ae + a_\mu \epsilon^\mu + a_{\alpha\beta} \epsilon^{\alpha\beta} + b_\mu \varepsilon^\mu + b \epsilon^5, \quad a, a_\mu, a_{\alpha\beta}, b_\mu, b \in \mathbb{C}. \quad (2.49)$$

We also have the isomorphism (2.43):

$$\mathbb{C}\mathcal{C}\ell(4) = \mathbb{C}\mathcal{C}\ell(2) \otimes_{\mathbb{C}} \mathbb{C}\mathcal{C}\ell(2). \quad (2.50)$$

Let  $\{e^1, e^2\}$  be generating elements of a complex Clifford algebra  $\mathbb{C}\mathcal{C}\ell(2)$  which obeys the relations (2.36). Then for instance, one can choose the generating elements

$$\epsilon^0 = e^1 \otimes e, \quad \epsilon^1 = ie^1 e^2 \otimes e^2, \quad \epsilon^2 = ie^1 e^2 \otimes e^1, \quad \epsilon^3 = e^2 \otimes e, \quad (2.51)$$

of a complex Clifford algebra  $\mathbb{C}\mathcal{C}\ell(4)$ . With the generating elements (2.51), the isomorphism (2.50) takes a form

$$\begin{aligned} \epsilon^{01} &= \frac{i}{2} e^2 \otimes e^2, & \epsilon^{02} &= \frac{i}{2} e^2 \otimes e^1, & \epsilon^{03} &= \frac{1}{2} e^1 e^2 \otimes e, \\ \epsilon^{12} &= -\frac{1}{2} e \otimes e^1 e^2, & \epsilon^{13} &= \frac{i}{2} e^1 \otimes e^2, & \epsilon^{23} &= \frac{i}{2} e^1 \otimes e^1, \\ \varepsilon^0 &= e^2 \otimes e^1 e^2, & \varepsilon^1 &= ie \otimes e^1, & \varepsilon^2 &= -ie \otimes e^2, & \varepsilon^3 &= e^1 \otimes e^1 e^2, \\ \epsilon^5 &= -e^1 e^2 \otimes e^1 e^2. \end{aligned} \quad (2.52)$$

□

**DEFINITION 2.6:** By a representation of a complex Clifford algebra  $\mathbb{C}\mathcal{C}\ell(n)$  is meant its morphism  $\rho$  to a complex algebra of linear endomorphisms of a finite-dimensional complex vector space. □

The following is a Corollary of Theorem 2.9.

**THEOREM 2.12:** If  $n$  is even, an exact irreducible representation of a complex Clifford algebra  $\mathbb{C}\mathcal{C}\ell(n)$  is unique up to an equivalence [9]. If  $n$  is odd there exist



two inequivalent exact irreducible representations of a complex Clifford algebra  $\mathbb{C}\mathcal{C}\ell(n)$ .  $\square$

**Remark 2.13:** Throughout the work, by representations of real and complex Clifford algebras are meant their exact representations only.  $\square$

In view of Corollary 2.10 and Theorem 2.12, we hereafter focus our consideration on real and complex Clifford algebras modelled over even vector spaces, and describe Clifford algebras of odd dimension as even subrings of those of even dimension (Lemmas 2.4 and 2.11, Example 2.8).

### 3 Automorphisms of Clifford algebras

We consider both generic ring automorphisms of a Clifford algebra and its automorphisms which preserve a specified generating space.

#### 3.1 Automorphisms of real Clifford algebras

Let  $\mathcal{C}\ell(V, \eta)$  be a real Clifford algebra modelled over an even-dimensional pseudo-Euclidean space  $(V, \eta)$ . By  $\text{Aut}[\mathcal{C}\ell(V, \eta)]$  is denoted the group of automorphisms of a real ring  $\mathcal{C}\ell(V, \eta)$ . A key point is the following.

**THEOREM 3.1:** Any automorphism of a real ring  $\mathcal{C}\ell(V, \eta)$  is inner.  $\square$

*Proof:* Theorem 2.5 states that any real Clifford algebra  $\mathcal{C}\ell(p, q)$ ,  $p - q = 0 \pmod 2$  as a ring is isomorphic to some matrix algebra  $\text{Mat}(m, \mathcal{K})$ ,  $\mathcal{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ . Such an algebra is simple. Algebras  $\text{Mat}(m, \mathcal{K})$ ,  $\mathcal{K} = \mathbb{R}, \mathbb{H}$ , are central simple real algebras with the center  $\mathcal{Z} = \mathbb{R}$ . Algebras  $\text{Mat}(m, \mathbb{C})$  are central simple complex algebras with the center  $\mathcal{Z} = \mathbb{C}$ . In accordance with the well-known Skolem–Noether theorem automorphisms of these algebras are inner.  $\square$

**THEOREM 3.2:** Invertible elements of a Clifford algebra  $\mathcal{C}\ell(V, \eta) = \text{Mat}(m, \mathcal{K})$  constitute a general linear matrix group

$$\mathcal{G}\mathcal{C}\ell(V, \eta) = \text{Gl}(m, \mathcal{K}). \quad (3.1)$$

$\square$

In particular, this group contains all elements  $v \in V \subset \mathcal{C}\ell(V, \eta)$  such that  $\eta(v, v) \neq 0$ . Acting in  $\mathcal{C}\ell(V, \eta)$  by left and right multiplications, the group  $\mathcal{G}\mathcal{C}\ell(V, \eta)$  (3.1) also acts in a Clifford algebra by the adjoint representation

$$\widehat{g} : q \rightarrow gqg^{-1}, \quad g \in \mathcal{G}\mathcal{C}\ell(V, \eta), \quad q \in \mathcal{C}\ell(V, \eta). \quad (3.2)$$

By virtue of Theorem 3.1, this representation provides an epimorphism

$$\zeta : \mathcal{G}\mathcal{C}\ell(V, \eta) = \text{Gl}(m, \mathcal{K}) \rightarrow \text{Gl}(m, \mathcal{K})/\mathcal{Z} = \text{Aut}[\mathcal{C}\ell(V, \eta)]. \quad (3.3)$$

Thus, we come to the following.

**THEOREM 3.3:** The group of automorphisms of a real Clifford algebra  $\mathcal{C}\ell(V, \eta) = \text{Mat}(m, \mathcal{K})$ ,  $\mathcal{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ , is a projective linear group

$$\text{Aut}[\mathcal{C}\ell(V, \eta)] = PGL(m, \mathcal{K}) = GL(m, \mathcal{K})/\mathcal{Z}, \quad (3.4)$$

where  $\mathcal{Z} = \mathbb{R}$  if  $\mathcal{K} = \mathbb{R}, \mathbb{H}$  and  $\mathcal{Z} = \mathbb{C}$  if  $\mathcal{K} = \mathbb{C}$ .  $\square$

Any ring automorphism  $g$  of  $\mathcal{C}\ell(V, \eta)$  sends a generating pseudo-Euclidean space  $(V, \eta)$  of  $\mathcal{C}\ell(V, \eta)$  onto an isometrically isomorphic pseudo-Euclidean space  $(V', \eta')$  such that

$$2\eta'(g(v), g(v'))e = g(v)g(v') + g(v')g(v) = 2\eta(v, v')e, \quad v, v' \in V.$$

It also is a generating space of a ring  $\mathcal{C}\ell(V, \eta)$ . Conversely, let  $(V, \eta)$  and  $(V', \eta')$  be two different pseudo-Euclidean generating spaces of the same signature of a ring  $\mathcal{C}\ell(V, \eta)$ . In accordance with Lemma 2.1, their isometric isomorphism  $(V, \eta) \rightarrow (V', \eta')$  gives rise to an automorphism of a ring  $\mathcal{C}\ell(V, \eta)$  which also is an isomorphism of Clifford algebras  $\mathcal{C}\ell(V, \eta) \rightarrow \mathcal{C}\ell(V', \eta')$ .

In particular, any (isometric) automorphism

$$g : V \ni v \rightarrow g(v) \in V, \quad \eta(g(v), g(v')) = \eta(v, v'), \quad g \in O(V, \eta),$$

of a pseudo-Euclidean generating space  $(V, \eta)$  is prolonged to an automorphism of a ring  $\mathcal{C}\ell(V, \eta)$  which also is an automorphism of a Clifford algebra  $\mathcal{C}\ell(V, \eta)$ . Then we have a monomorphism

$$O(V, \eta) \rightarrow \text{Aut}[\mathcal{C}\ell(V, \eta)]. \quad (3.5)$$

of a group  $O(V, \eta)$  of automorphisms of a pseudo-Euclidean space  $(V, \eta)$  to a group of ring automorphisms of  $\mathcal{C}\ell(V, \eta)$ . Herewith, an automorphism  $g \in O(V, \eta)$  of a ring  $\mathcal{C}\ell(V, \eta)$  is the identity one iff its restriction to  $V$  is an identity map of  $V$ . Consequently, the following is true.

**THEOREM 3.4:** A subgroup  $O(V, \eta) \subset \text{Aut}[\mathcal{C}\ell(V, \eta)]$  (3.5) exhausts all automorphisms of a ring  $\mathcal{C}\ell(V, \eta)$  which are automorphisms of a Clifford algebra  $\mathcal{C}\ell(V, \eta)$ .  $\square$

**Remark 3.1:** Elements of  $O(V, \eta)$  are represented by inner automorphisms of  $\mathcal{C}\ell(V, \eta)$  as follows. Given an element  $w \in V$ ,  $\eta(w, w) \neq 0$ , let

$$w^\perp = \{v \in V; \eta(v, w) = 0\}$$

be a hyperplane in  $V$  which is pseudo-orthogonal to  $w$  with respect to a metric  $\eta$ . Then any element  $v \in V$  is represented by a sum

$$v = u + \frac{\eta(v, w)}{\eta(w, w)}w, \quad u \in w^\perp.$$

Let us consider the inner automorphism  $\widehat{w}$  (3.2). Its restriction to  $V$  reads

$$\begin{aligned}\widehat{w} : V \ni v &\rightarrow wvw^{-1} = -v + 2\frac{\eta(w, v)}{\eta(w, w)}w \in V, \\ \eta(wvw^{-1}, wvw^{-1}) &= \eta(v, v).\end{aligned}\tag{3.6}$$

It is an automorphism of  $(V, \eta)$ . The transformation (3.6) is a composition of the total reflection  $v \rightarrow -v$  of  $V$  and a pseudo-orthogonal reflection

$$v \rightarrow v - 2\frac{\eta(w, v)}{\eta(w, w)}w\tag{3.7}$$

across a hyperplane  $w^\perp$ . Since  $(-w)^\perp = w^\perp$ , a pseudo-orthogonal reflection across a hyperplane  $w^\perp$  coincides with that across a hyperplane  $(-w)^\perp$ . Therefore, the total reflection of  $V$  commutes with the pseudo-orthogonal reflection (3.7) of  $V$  across a hyperplane and, as a consequence, with any inner automorphism  $\widehat{w}$  (3.6). It follows that any pseudo-orthogonal reflection (3.7) of  $V$  across a hyperplane is a composition of the total reflection of  $V$  and some inner automorphism (3.6) of  $V \subset \mathcal{Cl}(V, \eta)$ . Since a pseudo-Euclidean space  $V$  is of even dimension, its total reflection also is an inner automorphism

$$(\widehat{w}^1 \cdots \widehat{w}^n)(v) = (w^1 \cdots w^n)v(w^1 \cdots w^n)^{-1} = -v, \quad n = \dim V.\tag{3.8}$$

In this case, any pseudo-orthogonal reflection (3.7) of  $V$  across a hyperplane is represented by some inner automorphism of  $V \subset \mathcal{Cl}(V, \eta)$ . By the well-known Cartan–Dieudonné theorem, every element of a pseudo-orthogonal group  $O(V, \eta)$  can be written as a composition of  $r \leq \dim V$  pseudo-orthogonal reflections (3.7) across hyperplanes in  $V$  and, consequently, as a composition of inner automorphisms of  $V$ . Its prolongation onto a ring  $\mathcal{Cl}(V, \eta)$  also is an inner automorphism.  $\square$

Remark 3.1 gives something more. Let us consider a subgroup  $\text{Cliff}(V, \eta) \subset \mathcal{GCl}(V, \eta)$  generated by all invertible elements of  $V \subset \mathcal{Cl}(V, \eta)$ . It is called the Clifford group.

**THEOREM 3.5:** The homomorphism  $\zeta$  (3.3) of a Clifford group  $\text{Cliff}(V, \eta)$  to  $\text{Aut}[\mathcal{Cl}(V, \eta)]$  is its epimorphism

$$\zeta : \mathcal{GCl}(V, \eta) \supset \text{Cliff}(V, \eta) \rightarrow O(V, \eta) \subset \text{Aut}[\mathcal{Cl}(V, \eta)].\tag{3.9}$$

onto  $O(V, \eta)$ .  $\square$

*Proof:* The transformation (3.6) is an automorphism of  $(V, \eta)$  and, consequently, an element of  $O(V, \eta)$ . Thus, the homomorphism  $\zeta$  (3.3) of a Clifford group  $\text{Cliff}(V, \eta)$  to  $\text{Aut}[\mathcal{Cl}(V, \eta)]$  factorizes through the homomorphism (3.9). Conversely, it follows from Remark 3.1 that any element of  $O(V, \eta)$  is a composition of inner automorphisms (3.6) and (3.8) which are yielded by elements of  $\text{Cliff}(V, \eta)$ . Consequently, the homomorphism (3.9) is an epimorphism.  $\square$

Due to the factorization (3.9), any ring automorphism  $\widehat{v}$ ,  $v \in \text{Cliff}(V, \eta)$ , of  $\mathcal{C}\ell(V, \eta)$  also is an automorphism of a Clifford algebra  $\mathcal{C}\ell(V, \eta)$ . However, if  $(V', \eta')$  is a different generating space of a ring  $\mathcal{C}\ell(V, \eta)$ , we have a different Clifford subgroup  $\text{Cliff}(V', \eta')$  of a group  $\mathcal{G}\mathcal{C}\ell(V, \eta)$ . Then a Clifford group  $\text{Cliff}(V', \eta')$  provides ring automorphisms of  $\mathcal{C}\ell(V, \eta)$ , but not automorphisms of a Clifford algebra  $\mathcal{C}\ell(V, \eta)$ .

**Example 3.2:** Let us consider a ring  $\mathcal{C}\ell(2, 0) = \text{Mat}(2, \mathbb{R})$  (2.15) possessing the Euclidean basis  $\{e^1, e^2\}$  (2.16). Its group of invertible elements (3.1) is  $GL(2, \mathbb{R})$ . Elements of this group reads

$$ae + be^1 + ce^2 + de^2e^1 = \begin{pmatrix} a+c & b+d \\ b-d & a-c \end{pmatrix}. \quad (3.10)$$

They constitute a four-dimensional real vector space with a basis  $\{e, e^1, e^2, e^2e^1\}$ . Its elements  $\{e^1, e^2, e^2e^1\}$  generate a three-dimensional pseudo-Euclidean subspace  $(W, \chi)$  of signature  $(++-)$  such that

$$ww' + w'w = 2\chi(w, w')e, \quad w, w' \in W.$$

Then any two-dimensional pseudo-Euclidean subspace  $V$  of  $W$  is a generating space of a ring  $\mathcal{C}\ell(2, 0)$ , and *vice versa*. The group (3.3) of automorphisms of a ring  $\mathcal{C}\ell(2, 0)$  is

$$\text{Aut}[\mathcal{C}\ell(2, 0)] = PGL(2, \mathbb{R}) = SL(2, \mathbb{R})/\mathbb{Z}_2 = SO(2, 1). \quad (3.11)$$

Any automorphism of a ring  $\mathcal{C}\ell(2, 0)$  is an automorphism of  $(W, \chi)$ . Here-with, different automorphisms of  $\mathcal{C}\ell(2, 0)$  yield the distinct ones of  $W$ . Consequently, there is a monomorphism  $\text{Aut}[\mathcal{C}\ell(2, 0)] \rightarrow O(2, 1)$ . However, reflections  $e^1 \rightarrow -e^1$ ,  $e^2 \rightarrow -e^2$  and  $e^2e^1 \rightarrow -e^2e^1$  of  $W$  fail to be ring automorphisms because they are identity automorphisms of some two-dimensional subspaces of  $W$ . Therefore, we have the monomorphism (3.11). Elements of  $SO(2, 1)$  are given by compositions of automorphisms

$$\widehat{M}_\alpha \begin{pmatrix} e^1 \\ e^2 \\ e^2e^1 \end{pmatrix} = \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^1 \\ e^2 \\ e^2e^1 \end{pmatrix}, \quad (3.12)$$

$$\widehat{M}_s \begin{pmatrix} e^1 \\ e^2 \\ e^2e^1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh s & \sinh s \\ 0 & \sinh s & \cosh s \end{pmatrix} \begin{pmatrix} e^1 \\ e^2 \\ e^2e^1 \end{pmatrix} \quad (3.13)$$

$$\widehat{T} \begin{pmatrix} e^1 \\ e^2 \\ e^2e^1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} e^1 \\ e^2 \\ e^2e^1 \end{pmatrix} \quad (3.14)$$

of  $W$ . Note that automorphisms  $M_\alpha$  (3.12) and  $T$  (3.14) constitute a subgroup  $O(2) \subset SO(2, 1)$  of automorphism of a Clifford algebra  $\mathcal{C}\ell(2, 0)$  possessing an Euclidean generating basis  $\{e^1, e^2\}$ . They are inner automorphisms (3.2) generated, e.g., by the elements(3.10):

$$M_\alpha = e \cos(\alpha/2) - e^2e^1 \sin(\alpha/2), \quad T = e^2, \quad (3.15)$$

of a group  $O(2, \mathbb{R}) \subset \text{Mat}(2, \mathbb{R})$ . It should be however emphasized that there is no monomorphism

$$\text{Aut}[\mathcal{Cl}(2, 0)] \supset O(2, \mathbb{R}) \rightarrow \text{Mat}(2, \mathbb{R}),$$

whereas there exists an epimorphism

$$\text{Mat}(2, \mathbb{R}) \supset O(2, \mathbb{R}) \rightarrow O(2, \mathbb{R}) \subset \text{Aut}[\mathcal{Cl}(2, 0)]. \quad (3.16)$$

□

**Example 3.3:** Let us consider the Clifford algebra  $\mathcal{Cl}(4, 0) = \text{Mat}(2, \mathbb{H})$  (2.20) whose generating Euclidean space  $V$  possesses the basis  $\{\epsilon^0, \epsilon^1, \epsilon^2, \epsilon^3\}$  (2.46). Its elements  $\{\epsilon^0, \epsilon^1, \epsilon^2, \epsilon^3, \epsilon^5\}$  (see the notation (2.47)) make up a basis for a five-dimensional Euclidean space  $(W, \chi)$  such that

$$ww' + w'w = 2\chi(w, w')e, \quad w, w' \in W.$$

Similarly to Example 3.2, one can show that

$$\text{Aut}[\mathcal{Cl}(4, 0)] = PGL(2, \mathbb{H}) = SO(5).$$

Due to the isomorphisms (2.20), this also is the case of real rings  $\mathcal{Cl}(1, 3)$  and  $\mathcal{Cl}(0, 4)$ . □

**Example 3.4:** Let us consider the Clifford algebra  $\mathcal{Cl}(3, 1) = \text{Mat}(4, \mathbb{H}R)$  (2.25). Its automorphism group is

$$\text{Aut}[\mathcal{Cl}(3, 1)] = PGL(4, \mathbb{R}) = SL(4, \mathbb{R})/\mathbb{Z}_4. \quad (3.17)$$

□

### 3.2 Pin and Spin groups

The epimorphism (3.9) yields an action of a Clifford group  $\text{Cliff}(V, \eta)$  in a pseudo-Euclidean space  $(V, \eta)$  by the adjoint representation (3.2). However, this action is not effective. Therefore, one consider subgroups  $\text{Pin}(V, \eta)$  and  $\text{Spin}(V, \eta)$  of  $\text{Cliff}(V, \eta)$ . The first one is generated by elements  $v \in V$  such that  $\eta(v, v) = \pm 1$ . A group  $\text{Spin}(V, \eta)$  is defined as an intersection

$$\text{Spin}(V, \eta) = \text{Pin}(V, \eta) \cap \mathcal{Cl}^0(V, \eta) \quad (3.18)$$

of a group  $\text{Pin}(V, \eta)$  and the even subring  $\mathcal{Cl}^0(V, \eta)$  of a Clifford algebra  $\mathcal{Cl}(V, \eta)$ . In particular, generating elements  $v \in V$  of  $\text{Pin}(V, \eta)$  do not belong to its subgroup  $\text{Spin}(V, \eta)$ . Their images under the epimorphism  $\zeta$  (3.9) are reflections (3.6) of  $V$ .

**THEOREM 3.6:** The epimorphism (3.9) restricted to the  $\text{Pin}$  and  $\text{Spin}$  groups leads to short exact sequences of groups

$$e \rightarrow \mathbb{Z}_2 \longrightarrow \text{Pin}(V, \eta) \xrightarrow{\zeta} O(V, \eta) \rightarrow e. \quad (3.19)$$

$$e \rightarrow \mathbb{Z}_2 \longrightarrow \text{Spin}(V, \eta) \xrightarrow{\zeta} SO(V, \eta) \rightarrow e, \quad (3.20)$$

where  $\mathbb{Z}_2 \rightarrow (e, -e) \subset \text{Spin}(V, \eta)$ .  $\square$

It should be emphasized that an epimorphism  $\zeta$  in (3.19) and (3.20) is not a trivial bundle unless  $\eta$  is of signature  $(1, 1)$  (see Example 3.6). It is a universal coverings over each component of  $O(V, \eta)$ .

**Example 3.5:** Let us consider the Clifford algebra  $\mathcal{C}\ell(2, 0) = \text{Mat}(2, \mathbb{R})$  (2.15) possessing the Euclidean basis  $\{e^1, e^2\}$  (2.16) (Example 3.2). Then a group  $\text{Pin}(2, 0)$  is generated by elements

$$a_1 e^1 + a_2 e^2, \quad a_1^2 + a_2^2 = -\det(a_1 e^1 + a_2 e^2) = 1, \quad a_1, a_2 \in \mathbb{R}.$$

An even subring of  $\mathcal{C}\ell(2, 0)$  is represented by matrices  $ae + b\tau^2$ ,  $a, b \in \mathbb{R}$ . Then a group  $\text{Spin}(2, 0)$  consists of elements

$$a\mathbf{1} + be^2 e^1, \quad \det(ae + be^2 e^1) = a^2 + b^2 = 1, \quad a, b \in \mathbb{R},$$

i.e., of matrices  $M_\alpha$  (3.15) which constitute a group  $SO(2, \mathbb{R})$ . The epimorphism (3.20) reads

$$\text{Mat}(2, \mathbb{R}) \supset SO(2, \mathbb{R}) \rightarrow SO(2, \mathbb{R}) \subset \text{Aut}[\mathcal{C}\ell(2, 0)]. \quad (3.21)$$

(cf. (3.16)). Its kernel  $\zeta^{-1}(e)$  is a subgroup  $(e, -e)$  of  $SO(2, \mathbb{R})$ .  $\square$

**Example 3.6:** Let us consider the Clifford algebra  $\mathcal{C}\ell(1, 1) = \text{Mat}(2, \mathbb{R})$  (2.15). Its generating pseudo-Euclidean space possesses a basis  $\{e^1, e^1 e^2\}$ . Then a group  $\text{Pin}(1, 1)$  is generated by elements

$$a_1 e^1 + a_2 e^1 e^2, \quad a_1^2 - a_2^2 = \det(a_1 e^1 + a_2 e^1 e^2) = \pm 1, \quad a_1, a_2 \in \mathbb{R}.$$

An even subring of  $\mathcal{C}\ell(1, 1)$  is represented by matrices  $ae + be^2$ ,  $a, b \in \mathbb{R}$ . Then a group  $\text{Spin}(1, 1)$  consists of elements

$$ae + be^2, \quad \det(ae + be^2) = a^2 - b^2 = \pm 1, \quad a, b \in \mathbb{R},$$

i.e., of elements

$$\pm[(\cosh s)e + (\sinh s)e^2] = \pm \exp(se^2) = \pm M_s, \quad \pm e^2 M_s.$$

It is isomorphic to a group  $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{R}^+$ , where  $\mathbb{R}^+$  is a group of positive real numbers. Its epimorphism  $\zeta$  (3.20) onto a subgroup  $SO(1, 1)$  of  $\text{Aut}[\mathcal{C}\ell(2, 0)] = SO(2, 1)$  has the kernel  $(e, -e) = \mathbb{Z}_2 \times e \times e$ . It is readily observed that  $\zeta(e^2)$  is a total reflection of  $\mathbb{R}^2$ . Therefore the exact sequence (3.20) for  $\text{Spin}(1, 1)$  is reduced to the exact sequence

$$e \rightarrow \mathbb{Z}_2 \longrightarrow \text{Spin}^+(1, 1) \xrightarrow{\zeta} SO^0(1, 1) \rightarrow e$$

where  $\text{Spin}^+(1, 1) = \mathbb{Z}_2 \times \mathbb{R}^+$  is a subgroup of matrices  $\pm M_s$  and  $SO^0(1, 1)$  is a connected component of the unit of  $SO(1, 1)$ .  $\square$

**Example 3.7:** Let the Clifford algebra  $\mathcal{Cl}(1, 3) = \text{Mat}(2, \mathbb{H})$  (2.20) be represented as a subalgebra of the complex Clifford algebra  $\mathbb{C}\mathcal{Cl}(4) = \text{Mat}(4, \mathbb{C})$  (2.45) whose generating elements are Dirac's matrices  $(\tilde{\gamma}^0, \tilde{\gamma}^j)$  (2.23). A group  $\text{Pin}(1, 3)$  is generated by matrices

$$a_\mu \tilde{\gamma}^\mu, \quad a_0^2 - \sum_{i=1,2,3} a_i^2 = \det(a_0 \mathbf{1} + a_i \sigma^i) = \pm 1, \quad a_\mu \in \mathbb{R}, \quad (3.22)$$

$$\det(a_\mu \tilde{\gamma}^\mu) = \left( a_0^2 - \sum_{i=1,2,3} a_i^2 \right)^2 = 1.$$

A group  $\text{Spin}(1, 3)$  is a subgroup of  $\text{Pin}(1, 3)$  whose elements are even products of the matrices (3.22). It is generated by matrices

$$(a_0 \tilde{\gamma}^0 + a_i \tilde{\gamma}^i)(b_0 \tilde{\gamma}^0 + b_i \tilde{\gamma}^i) = (a_0 \mathbf{1} + a_i \tilde{\gamma}^i \tilde{\gamma}^0)(b_0 \mathbf{1} + b_i \tilde{\gamma}^0 \tilde{\gamma}^i), \quad a_\mu, b_\mu \in \mathbb{R},$$

$$\det(a_0 \mathbf{1} + a_i \sigma^i) = \pm 1, \quad \det(b_0 \mathbf{1} + b_i \sigma^i) = \pm 1,$$

$$\det(a_\mu \tilde{\gamma}^\mu) = \det(b_\mu \tilde{\gamma}^\mu) = 1.$$

Then elements of  $\text{Spin}(1, 3)$  take a form

$$\begin{pmatrix} c_0 \mathbf{1} + c_i \sigma^i & 0 \\ 0 & \bar{c}_0 \mathbf{1} - \bar{c}_i \sigma^i \end{pmatrix}, \quad c_\mu \in \mathbb{C},$$

$$\det(c_0 \mathbf{1} + c_i \sigma^i) = \det(\bar{c}_0 \mathbf{1} - \bar{c}_i \sigma^i) = \pm 1.$$

They read

$$M_A = \begin{pmatrix} A & 0 \\ 0 & \text{Tr} A^* \mathbf{1} - A^* \end{pmatrix}, \quad (3.23)$$

where  $A$  are complex  $(2 \times 2)$ -matrices such that

$$\det A = \det(\text{Tr} A^* \mathbf{1} - A^*) = \pm 1.$$

A group  $\text{Spin}(1, 3)$  contains two connected components  $\text{Spin}^+(1, 3)$  and  $\text{Spin}^-(1, 3)$  which consist of the elements (3.23) with  $\det A = 1$  and  $\det A = -1$ , respectively. Being a connected component of the unity, the first one is a group  $SL(2, \mathbb{C})$ . Elements of  $\text{Spin}^-(1, 3)$  come from elements of  $\text{Spin}^+(1, 3)$  by means of multiplication

$$\text{Spin}^+(1, 3) \ni M_A \rightarrow M_{i\mathbf{1}} M_A \in \text{Spin}^-(1, 3).$$

We have the exact sequence (3.20):

$$e \rightarrow \mathbb{Z}_2 \longrightarrow \text{Spin}(1, 3) \xrightarrow{\zeta} SO(1, 3) \rightarrow e, \quad (3.24)$$

where  $\zeta(M_{i\mathbf{1}}) \in SO(1, 3)$  is a total reflection. This exact sequence is restricted to the exact sequence

$$e \rightarrow \mathbb{Z}_2 \longrightarrow \text{Spin}^+(1, 3) \xrightarrow{\zeta} SO^0(1, 3) \rightarrow e \quad (3.25)$$

where  $SO^0(1, 3)$ , called the proper Lorentz group, is a connected component of the unit of  $SO(1, 3)$ . Let us denote

$$L_s = \text{Spin}^+(1, 3) = SL(2, \mathbb{C}), \quad L = SO^0(1, 3). \quad (3.26)$$

Group spaces of  $L_s$  and  $L$  are topological spaces  $S^3 \times \mathbb{R}^3$  and  $\mathbb{R}P^3 \times \mathbb{R}^3$ , respectively.  $\square$

**Example 3.8:** Let the Clifford algebra  $\mathcal{Cl}(4, 0) = \text{Mat}(2, \mathbb{H})$  (2.20) be represented as a subalgebra of the complex Clifford algebra  $\mathbb{C}\mathcal{Cl}(4) = \text{Mat}(4, \mathbb{C})$  (2.45) whose generating elements are the matrices  $(\tilde{\gamma}^0, -i\tilde{\gamma}^j)$  (2.23). A group  $\text{Pin}(4, 0)$  is generated by matrices

$$a_0\tilde{\gamma}^0 + ia_i\tilde{\gamma}^i, \quad \sum_{\mu=0,\dots,3} a_\mu^2 = \det(a_0\mathbf{1} + a_i\tau^i) = 1, \quad a_\mu \in \mathbb{R}, \quad (3.27)$$

$$\det(a_0\tilde{\gamma}^0 + ia_i\tilde{\gamma}^i) = \left( \sum_{\mu=0,1,2,3} a_\mu^2 \right)^2 = 1.$$

A group  $\text{Spin}(4, 0)$  is a subgroup of  $\text{Pin}(4, 0)$  whose elements are even products of the matrices (3.27). It is generated by matrices

$$(a_0\tilde{\gamma}^0 + ia_i\tilde{\gamma}^i)(b_0\tilde{\gamma}^0 + ib_i\tilde{\gamma}^i) = (a_0\mathbf{1} + ia_i\tilde{\gamma}^i\tilde{\gamma}^0)(b_0\mathbf{1} + ib_i\tilde{\gamma}^0\tilde{\gamma}^i), \quad a_\mu, b_\mu \in \mathbb{R},$$

$$\det(a_0\mathbf{1} + a_i\tau^i) = \det(b_0\mathbf{1} + b_i\tau^i) = 1,$$

$$\det(a_0\tilde{\gamma}^0 + ia_i\tilde{\gamma}^i) = \det(b_0\tilde{\gamma}^0 + ib_i\tilde{\gamma}^i) = 1,$$

which take a form

$$\begin{pmatrix} c_0\mathbf{1} + c_i\tau^i & 0 \\ 0 & d_0\mathbf{1} + d_i\tau^i \end{pmatrix}, \quad c_\mu, d_\mu \in \mathbb{R},$$

$$\det(c_0\mathbf{1} + c_i\tau^i) = \det(d_0\mathbf{1} + d_i\tau^i) = 1.$$

Then elements of  $\text{Spin}(4, 0)$  read

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad \det A = \det B = 1, \quad (3.28)$$

where  $A, B$  are unimodular unitary complex  $(2 \times 2)$ -matrices. Thus, a group  $\text{Spin}(4, 0)$  is isomorphic to a product  $SU(2) \times SU(2)$ . It contains a subgroup  $(e, \gamma^0)$  such that  $\zeta(\gamma^0)$  is a total reflection of  $\mathbb{R}^4$ . Thus, the exact sequence (3.20) is reduced

$$e \rightarrow \mathbb{Z}_2 \longrightarrow \text{Spin}^+(4, 0) \xrightarrow{\zeta} SO(3) \times SO(3) \rightarrow e \quad (3.29)$$

where  $\text{Spin}^+(4, 0) = \mathbb{Z}_2 \times SU(2)/\mathbb{Z}_2 \times SU(2)/\mathbb{Z}_2$  and  $\mathbb{Z}_2 = (e, -e)$ .  $\square$



### 3.3 Automorphisms of complex Clifford algebras

Let  $\mathbb{C}\mathcal{C}\ell(n)$  be the complex Clifford algebra (2.35) of even  $n$ .

**THEOREM 3.7:** All automorphisms of a complex Clifford algebra  $\mathbb{C}\mathcal{C}\ell(n)$  are inner.  $\square$

*Proof:* By virtue of Theorem 2.9, there is the ring isomorphism (2.43):

$$\mathbb{C}\mathcal{C}\ell(n) = \text{Mat}(2^{n/2}, \mathbb{C}). \quad (3.30)$$

In accordance with Corollary 2.10, this algebra is a central simple complex algebra with the center  $\mathcal{Z} = \mathbb{C}$ . In accordance with the above-mentioned Skolem-Noether theorem automorphisms of these algebras are inner.  $\square$

**THEOREM 3.8:** Invertible elements of the Clifford algebra (3.30) constitute a general linear group  $GL(2^{n/2}, \mathbb{C})$ .  $\square$

**THEOREM 3.9:** Acting in  $\mathbb{C}\mathcal{C}\ell(n)$  by left and right multiplications, this group also acts in a Clifford algebra by the adjoint representation, and we obtain an epimorphism

$$GL(2^{n/2}, \mathbb{C}) \rightarrow PGL(2^{n/2}, \mathbb{C}) = \text{Aut}[\mathcal{C}\ell(n)].$$

and a group of its automorphisms is a projective linear group

$$\begin{aligned} \text{Aut}[\mathcal{C}\ell(n)] &= PGL(2^{n/2}, \mathbb{C}) = GL(2^{n/2}, \mathbb{C})/\mathbb{C} = \\ &= SL(2^{n/2}, \mathbb{C})/\mathbb{Z}_{2^{n/2}}. \end{aligned} \quad (3.31)$$

$\square$

Automorphisms of its real subrings  $\mathcal{C}\ell(n, 0)$  yield automorphisms of  $\mathbb{C}\mathcal{C}\ell(n)$ , but do not exhaust all automorphisms of  $\mathbb{C}\mathcal{C}\ell(n)$  (Theorem 3.10).

Let us note that automorphisms under discussions need not be automorphisms of  $\mathbb{C}\mathcal{C}\ell(n)$  as an involutive algebra (Remark 3.10)

Any automorphism  $g$  of a complex Clifford algebra  $\mathbb{C}\mathcal{C}\ell(n)$  sends its Euclidean generating space  $(\mathcal{V}, \kappa)$  onto some generating space

$$(\mathcal{V}', \kappa'), \quad \kappa'(g(v), g(v')) = \kappa(v, v'), \quad v, v' \in \mathcal{V},$$

which is the Euclidean one with respect to the basis  $\{g(e^i)\}$ . If  $g$  preserves  $\mathcal{V}$ , then

$$\kappa(g(v), g(v')) = \kappa(v, v'), \quad v, v' \in \mathcal{V},$$

i.e.,  $g$  is an automorphism of a metric space  $(\mathcal{V}, \kappa)$ .

Conversely, any automorphism

$$g : \mathcal{V} \ni v \rightarrow gv \in \mathcal{V}, \quad \kappa(g(v), g(v')) = \kappa(v, v'), \quad g \in O(n, \mathbb{C}),$$

of an Euclidean generating space  $(\mathcal{V}, \kappa)$  is prolonged to an automorphism of a ring  $\mathbb{C}\mathcal{C}\ell(n)$ . Then we have a monomorphism

$$O(n, \mathbb{C}) \rightarrow \text{Aut}[\mathbb{C}\mathcal{C}\ell(n)] \quad (3.32)$$

of a group  $O(n, \mathbb{C})$  of automorphisms of an Euclidean generating space  $(\mathcal{V}, \kappa)$  to a group of ring automorphisms of  $\mathbb{C}\mathcal{C}\ell(n)$ . Herewith, an automorphism  $g \in O(n, \mathbb{C})$  of a complex ring  $\mathbb{C}\mathcal{C}\ell(n)$  is the identity one iff its restriction to  $\mathcal{V}$  is an identity map of  $\mathcal{V}$ . Consequently, the following is true.

**THEOREM 3.10:** All ring automorphisms of a complex Clifford algebra  $\mathbb{C}\mathcal{C}\ell(n)$  preserving its Euclidean generating space constitute a group  $O(n, \mathbb{C})$ .  $\square$

Let  $Z\mathbb{C}\mathcal{C}\ell(n)$  denote a set of Euclidean generating spaces of a complex Clifford algebra  $\mathbb{C}\mathcal{C}\ell(n)$ . If  $n > 1$ , a set  $Z\mathbb{C}\mathcal{C}\ell(n)$  contains more than one element. Indeed, let  $\mathcal{V}$  be a generating space of  $\mathbb{C}\mathcal{C}\ell(n)$  spanned by its Euclidean basis  $\{e^1, \dots, e^n\}$ . Then,  $\{e^1, ie^1e^2, \dots, ie^1e^n\}$  is an Euclidean basis for a different generating space of  $\mathbb{C}\mathcal{C}\ell(n)$ .

**LEMMA 3.11:** A group  $\text{Aut}[\mathbb{C}\mathcal{C}\ell(n)]$  of ring automorphism of a complex Clifford algebra  $\mathbb{C}\mathcal{C}\ell(n)$  acts in a set  $Z\mathbb{C}\mathcal{C}\ell(n)$  effectively and transitively, i.e., no element  $\text{Aut}[\mathbb{C}\mathcal{C}\ell(n)] \ni g \neq e$  is the identity morphism of  $Z\mathbb{C}\mathcal{C}\ell(n)$  and, for any two different elements of  $Z\mathbb{C}\mathcal{C}\ell(n)$ , there exists a ring automorphism of  $\mathbb{C}\mathcal{C}\ell(n)$  which sends them onto each other.  $\square$

*Proof:* Let an automorphism  $\text{Aut}[\mathbb{C}\mathcal{C}\ell(n)] \ni g \neq e$  preserves some Euclidean generating space  $(\mathcal{V}, \kappa)$  of  $\mathbb{C}\mathcal{C}\ell(n)$ . There exists an element  $v \in \mathcal{V}$  such that  $v^2 = e$  and  $g(v) \neq v$ . Then  $\mathcal{V}$  admits an Euclidean basis  $\{v, e^2, \dots, e^n\}$ , and there exists a different generating space of  $\mathbb{C}\mathcal{C}\ell(n)$  possessing an Euclidean basis  $\{v, ve^2, \dots, ve^n\}$ . It is not preserved by an automorphism  $g$ . Let  $\mathcal{V}$  and  $\mathcal{V}'$  be two different generating spaces of  $\mathbb{C}\mathcal{C}\ell(n)$  with Euclidean bases  $\{e^1, \dots, e^n\}$  and  $\{e'^1, \dots, e'^n\}$ , respectively. Then an association  $e^i \rightarrow e'^i$  provides an isomorphism  $\mathcal{V} \rightarrow \mathcal{V}'$  which is extended to an automorphism of  $\mathbb{C}\mathcal{C}\ell(n)$ .  $\square$

It follows from Theorem 3.10 and Lemma 3.11 that, if  $n > 1$ , a set  $Z\mathbb{C}\mathcal{C}\ell(n)$  of generating spaces of a complex Clifford algebra  $\mathbb{C}\mathcal{C}\ell(n)$  is a homogeneous space

$$Z\mathbb{C}\mathcal{C}\ell(n) = PGL(2^{n/2}, \mathbb{C})/O(n, \mathbb{C}). \quad (3.33)$$

Given a complex Clifford algebra  $\mathbb{C}\mathcal{C}\ell(n)$ , let  $\mathcal{C}\ell(m, n-m)$  be a real Clifford algebra. Due to the canonical ring monomorphism  $\mathcal{C}\ell(m, n-m) \rightarrow \mathbb{C}\mathcal{C}\ell(n)$  (2.38), there is the canonical group monomorphism

$$\mathcal{G}\mathcal{C}\ell(m, n-m) \rightarrow \text{Mat}(2^{n/2}, \mathbb{C}). \quad (3.34)$$

Since all ring automorphisms of a Clifford algebra are inner (Theorem 3.1), they are extended to inner automorphisms of a complex Clifford algebra  $\mathbb{C}\mathcal{C}\ell(n)$  and, consequently, there is a group monomorphism

$$\text{Aut}[\mathcal{C}\ell(m, n-m)] \rightarrow PGL(2^{n/2}, \mathbb{C}). \quad (3.35)$$

**Example 3.9:** Let us consider the complex Clifford algebra  $\mathbb{C}\mathcal{C}\ell(2) = \text{Mat}(2, \mathbb{C})$  (2.44). It possesses an Euclidean basis  $\{e^1, e^2\}$  obeying the relations (2.36). Its elements  $\{e^1, e^2, ie^1e^2\}$  form a basis for a three-dimensional complex subspace  $W$  of  $\mathbb{C}\mathcal{C}\ell(2)$  provided with a non-degenerate bilinear form  $\chi$  such that

$$ww' + w'w = 2\chi(w, w')e, \quad w, w' \in W.$$

Then any two-dimensional generating space of  $\mathbb{C}\mathcal{C}\ell(2)$  is a subspace of  $W$ , and any ring automorphism of  $\mathbb{C}\mathcal{C}\ell(2)$  is that of  $W$ . By virtue of Theorem 3.7, the group of automorphisms of  $\mathbb{C}\mathcal{C}\ell(2)$  is

$$\text{Aut}[\mathbb{C}\mathcal{C}\ell(2)] \rightarrow SL(2, \mathbb{C})/\mathbb{Z}_2 = SO(3, \mathbb{C}). \quad (3.36)$$

Elements of  $SO(3, \mathbb{C})$  are given by compositions of automorphisms

$$\widehat{a}_{\phi s} \begin{pmatrix} e^1 \\ e^2 \\ e^1e^2 \end{pmatrix} = \begin{pmatrix} \cos(\phi + is) & -\sin(\phi + is) & 0 \\ \sin(\phi + is) & \cos(\phi + is) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^1 \\ e^2 \\ e^1e^2 \end{pmatrix}, \quad (3.37)$$

$$\widehat{a}_{\theta r} \begin{pmatrix} e^1 \\ e^2 \\ e^1e^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta + ir) & i\sin(\theta + ir) \\ 0 & i\sin(\theta + ir) & \cos(\theta + ir) \end{pmatrix} \begin{pmatrix} e^1 \\ e^2 \\ e^1e^2 \end{pmatrix} \quad (3.38)$$

of  $W$ . In accordance with the relation (3.33) and the isomorphism (3.36), we obtain a set

$$Z\mathbb{C}\mathcal{C}\ell(2) = SO(3, \mathbb{C})/O(2, \mathbb{C}) \quad (3.39)$$

of generating spaces of a complex Clifford algebra  $\mathbb{C}\mathcal{C}\ell(2)$ . The automorphisms (3.37) – (3.38) are inner automorphisms

$$\widehat{a}_{\phi s}(a) = a_{\phi s} a a_{\phi s}^{-1}, \quad \widehat{a}_{\theta r}(a) = a_{\theta r} a a_{\theta r}^{-1}, \quad a \in \mathbb{C}\mathcal{C}\ell(2), \quad (3.40)$$

$$a_{\phi s} = e \cos(\phi/2 + is/2) + e^1 e^2 \sin(\phi/2 + is/2), \quad (3.41)$$

$$a_{\phi s}^{-1} = e \cos(\phi/2 + is/2) - e^1 e^2 \sin(\phi/2 + is/2),$$

$$a_{\theta r} = e \cos(\theta/2 + ir/2) + ie^1 \sin(\theta/2 + ir/2), \quad (3.42)$$

$$a_{\theta r}^{-1} = e \cos(\theta/2 + ir/2) - ie^1 \sin(\theta/2 + ir/2),$$

In particular, the Spin groups  $\text{Spin}(2, 0)$ ,  $\text{Spin}(0, 2)$  and  $\text{Spin}(1, 1)$  in Examples 3.5 and 3.6 yield inner automorphism  $\widehat{a}_{\phi, 0}$  and  $\widehat{a}_{0, s}$  (3.37) of  $\mathbb{C}\mathcal{C}\ell(2)$ , respectively. There are natural injections of a group  $SO(2) = \zeta(\text{Spin}(2, 0))$  of automorphisms of a Clifford algebra  $\mathcal{C}\ell(2, 0)$ , a group  $SO(1, 1) = \zeta(\text{Spin}(1, 1))$  of automorphisms of a Clifford algebra  $\mathcal{C}\ell(1, 1)$  and a group  $SO(2) = \zeta(\text{Spin}(0, 2))$  of automorphisms of a Clifford algebra  $\mathcal{C}\ell(0, 2)$  to  $SO(3, \mathbb{C})$ .  $\square$

**Remark 3.10:** Let us note that automorphisms  $\widehat{a}_{\phi s \neq 0}$  (3.37) and  $\widehat{a}_{\theta r \neq 0}$  (3.38) do not transform Hermitian elements to Hermitian elements, and thus they are not automorphisms of an involutive algebra  $\mathbb{C}\mathcal{C}\ell(2)$ .  $\square$

**Example 3.11:** Let us consider the complex Clifford algebra  $\mathbb{C}\mathcal{C}\ell(4) = \text{Mat}(4, \mathbb{C})$  (2.45) possessing an Euclidean basis  $\{\epsilon^\mu\}$ . Its elements  $\{\epsilon^\mu, \epsilon^5\}$  form

a basis for a five-dimensional complex subspace  $W$  of  $\mathbb{C}\mathcal{C}\ell(4)$  provided with a non-degenerate bilinear form  $\chi$  such that

$$ww' + w'w = 2\chi(w, w')e, \quad w, w' \in W.$$

Then any four-dimensional complex subspace  $\mathcal{V}$  of  $W$  provided with an induced bilinear form is a generating space of a complex Clifford algebra  $\mathbb{C}\mathcal{C}\ell(4)$ . By virtue of Theorem 3.7, the group of automorphisms of  $\mathbb{C}\mathcal{C}\ell(4)$  is

$$\text{Aut}[\mathbb{C}\mathcal{C}\ell(4)] = PGL(4, \mathbb{C}) = SO(6, \mathbb{C})/\mathbb{Z}_2. \quad (3.43)$$

Then in accordance with the relation (3.33), we obtain a set

$$Z\mathbb{C}\mathcal{C}\ell(4) = PGL(4, \mathbb{C})/O(4, \mathbb{C}) \quad (3.44)$$

of generating spaces of a complex Clifford algebra  $\mathbb{C}\mathcal{C}\ell(4)$ .  $\square$

## 4 Spinor spaces of complex Clifford algebras

As was mentioned above, we define spinor spaces in terms of Clifford algebras.

**DEFINITION 4.1:** A real spinor space  $\Psi(m, n - m)$  is defined as a carrier space of an irreducible representation of a Clifford algebra  $\mathcal{C}\ell(m, n - m)$ .  $\square$

It also carries out a representation of the corresponding group  $\text{Spin}(m, n - m) \subset \mathcal{C}\ell(m, n - m)$  [9].

If  $n$  is even, such a real spinor space is unique up to an equivalence in accordance with Theorem 2.7. However, Examples 2.4 – 2.5 of Clifford algebras  $\mathcal{C}\ell(0, 2)$  and  $\mathcal{C}\ell(2, 0)$ , respectively, show that spinor spaces  $\Psi(m, n - m)$  and  $\Psi(m', n - m')$  need not be isomorphic vector spaces for  $m' \neq m$ .

For instance, a Dirac spinor space is defined to be a spinor space  $\Psi(1, 3)$  of a Clifford algebra  $\mathcal{C}\ell(1, 3)$  (Example 2.6). It differs from a Majorana spinor space  $\Psi(3, 1)$  of a Clifford algebra  $\mathcal{C}\ell(3, 1)$  (Example 2.7). In contrast with the four-dimensional real matrix representation (2.26) of  $\mathcal{C}\ell(3, 1)$ , a representation of a real Clifford algebra  $\mathcal{C}\ell(3, 1)$  by complex Dirac's matrices (2.22) is not a representation a real Clifford algebra by virtue of Definition 2.3. From the physical viewpoint, Dirac spinor fields describing charged fermions are complex fields.

Therefore, we consider complex spinor spaces.

**DEFINITION 4.2:** A complex spinor space  $\Psi(n)$  is defined as a carrier space of an irreducible representation of a complex Clifford algebra  $\mathbb{C}\mathcal{C}\ell(n)$ .  $\square$

Since  $n$  is even, a representation  $\Psi(n)$  is unique up to an equivalence in accordance with Theorem 2.12. Therefore, it is sufficient to describe a complex spinor space  $\Psi(n)$  as a subspace of a complex Clifford algebra  $\mathbb{C}\mathcal{C}\ell(n)$  which acts on  $\Psi(n)$  by left multiplications.

Given a complex Clifford algebra  $\mathbb{C}\mathcal{C}\ell(n)$ , let us consider its non-zero minimal left ideal which  $\mathcal{C}\ell(n)$  acts on by left multiplications. It is a finite-dimensional

complex vector space. Therefore, an action of a complex Clifford algebra  $\mathbb{C}\mathcal{C}\ell(n)$  in a minimal left ideal by left multiplications defines a linear representation of  $\mathbb{C}\mathcal{C}\ell(n)$ . It obviously is irreducible. In this case, a minimal left ideal of  $\mathbb{C}\mathcal{C}\ell(n)$  is a complex spinor space  $\Psi(n)$ . Thus, we come to an equivalent definition of a complex spinor space.

DEFINITION 4.3: Complex spinor spaces  $\Psi(n)$  are minimal left ideals of a complex Clifford algebra  $\mathbb{C}\mathcal{C}\ell(n)$  which carry out its irreducible representation (1.2).  $\square$

By virtue of Theorem 2.9, there is a ring isomorphism  $\mathbb{C}\mathcal{C}\ell(n) = \text{Mat}(2^{n/2}, \mathbb{C})$  (3.30). Then we come to the following.

THEOREM 4.1: A spinor representation of a Clifford algebra  $\mathbb{C}\mathcal{C}\ell(n)$  is equivalent to the canonical representation of  $\text{Mat}(2^{n/2}, \mathbb{C})$  by matrices in a complex vector space  $\mathbb{C}^{2^{n/2}}$ , i.e.,  $\Psi(n) = \mathbb{C}^{2^{n/2}}$ .  $\square$

COROLLARY 4.2: A spinor space  $\Psi(n) \subset \mathbb{C}\mathcal{C}\ell(n)$  also carries out the left-regular irreducible representation of a group  $GL(2^{n/2}, \mathbb{C}) = \mathcal{G}\mathbb{C}\mathcal{C}\ell(n)$  which is equivalent to the natural matrix representation of  $GL(2^{n/2}, \mathbb{C})$  in  $\mathbb{C}^{2^{n/2}}$ .  $\square$

COROLLARY 4.3: Owing to the monomorphism  $\mathcal{C}\ell(m, n-m) \rightarrow \mathbb{C}\mathcal{C}\ell(n)$  (2.38), a spinor space  $\Psi(n)$  also carries out a representation of real Clifford algebras  $\mathcal{C}\ell(m, n-m)$ , their Clifford<sub>I</sub> Pin and Spin groups, though these representation need not be reducible.  $\square$

In order to describe complex spinor spaces in accordance with Definition 4.3, we are based on the following (Theorem 4.6).

LEMMA 4.4: If a minimal left ideal  $Q$  of a complex Clifford algebra  $\mathbb{C}\mathcal{C}\ell(n)$  is generated by an element  $q$ , then  $q^2 = \lambda q$ ,  $\lambda \in \mathbb{C}$ .  $\square$

*Proof:* Let  $q \in Q$  such that  $q^2 \neq \lambda q$ ,  $\lambda \in \mathbb{C}$ . There are two variants: (i)  $q^N = 0$  starting with some natural number  $N > 2$ , (ii) there is no  $m > 2$  such that  $q^m = 0$ . In the first case, let us consider a left ideal  $Q'$  generated by  $q^{N-1} = q^{N-2}q \in Q$ . It does not contains  $q$  because, if  $q = bq^{N-1}$ ,  $b \in \mathbb{C}\mathcal{C}\ell(n)$ , then  $q^2 = bq^N = 0$  that contradicts the condition  $q^2 \neq \lambda q$ . Thus, a left ideal  $Q'$  is a proper subset of  $Q$ , i.e.,  $Q$  fails to be minimal. In the second variant, since  $q^2 \neq \lambda q$  and  $Q$  is a finite-dimensional complex space, there exists a natural number  $m > 2$  such that elements  $q^r$ ,  $r = 2, \dots, m$ , are linearly dependent, i.e.,

$$\sum_{r=2}^m \lambda_r q^r = 0, \quad \lambda_r \in \mathbb{C}.$$

This equality is brought into the form  $q^p c = cq^p = 0$ ,  $c \in \mathbb{C}\mathcal{C}\ell(n)$  for some  $1 < p < m$ . Let us consider a left ideal  $Q'$  generated by an element  $cq^{p-1} = (cq^{p-2})q \in Q$ . It does not contains an element  $q$  because, if  $q = bcq^{p-1}$ ,  $b \in \mathbb{C}\mathcal{C}\ell(n)$ , then  $q^2 = bcq^p = 0$  that contradicts the condition  $q^2 \neq \lambda q$ . Thus, a left ideal  $Q'$  is a proper subset of  $Q$ , i.e.,  $Q$  fails to be minimal.  $\square$

Lemma 4.4 gives something more. Since a minimal left ideal  $Q$  of  $\mathbb{C}\mathcal{C}\ell(n)$  is generated by any its element, each element  $q \in Q$  possesses a property  $q^2 = \lambda q$ ,  $\lambda \in \mathbb{C}$ .

LEMMA 4.5: A minimal left ideal of a complex Clifford algebra  $\mathbb{C}\mathcal{C}\ell(n)$  contains a non-zero Hermitian element, and thus it is generated by a Hermitian element.  $\square$

*Proof:* Let  $q \neq 0$  be an element of  $Q$ . Then,  $q^*q \neq 0$  in accordance with the inequality (2.42), and this is a Hermitian element of  $Q$ .  $\square$

By virtue of Lemmas 4.4 – 4.5, any minimal left ideal of a complex Clifford algebra  $\mathbb{C}\mathcal{C}\ell(n)$  is generated by a Hermitian idempotent  $p = p^*$ ,  $p^2 = p$ . Of course, it is not invertible because invertible elements generate an algebra  $\mathbb{C}\mathcal{C}\ell(n)$  which contains proper left ideals. It is readily observed that any Hermitian idempotent takes a form

$$p = \frac{1}{2}(e + s), \quad s^2 = e, \quad s^* = s, \quad s \neq e. \quad (4.1)$$

Thus, the following has been proved.

THEOREM 4.6: Any complex spinor space  $\Psi(n)$  is generated by some Hermitian idempotent  $p \in \Psi(n)$  (4.1).  $\square$

The converse however need not be true.

**Example 4.1:** Let us consider a Hermitian idempotent  $p \in \text{Mat}(2^{n/2}, \mathbb{C})$  whose non-zero component is only  $p_{11} = 1$ . It generates a minimal left ideal  $\Psi_{11}(n) \subset \text{Mat}(2^{n/2}, \mathbb{C})$  which consists of matrices  $a \in \text{Mat}(2^{n/2}, \mathbb{C})$  whose columns, except  $a_{1i}$  equal zero.  $\square$

Certainly, an automorphism of a Clifford algebra  $\mathbb{C}\mathcal{C}\ell(n)$  sends a spinor space onto a spinor space.

LEMMA 4.7: An action of a group  $PGL(2^{n/2}, \mathbb{C})$  of automorphisms of  $\mathbb{C}\mathcal{C}\ell(n)$  in a set  $S\Psi(n)$  of spinor spaces is transitive.  $\square$

*Proof:* Let  $\Psi(n)$  and  $\Psi'(n)$  are spinor space defined by Hermitian idempotents  $p \in \text{Mat}(2^{n/2}, \mathbb{C})$  and  $p' \in \text{Mat}(2^{n/2}, \mathbb{C})$ . Since right-regular representation of a group  $GL(2^{n/2}, \mathbb{C})$  in  $\text{Mat}(2^{n/2}, \mathbb{C})$  is transitive, there exists an element  $g \in \text{Mat}(2^{n/2}, \mathbb{C})$  so that  $p' = pg^{-1}$ . Then a spinor space  $\Psi'(n)$  is generated by an idempotent  $gpg^{-1}$ , and an inner automorphism  $a \rightarrow gpg^{-1}a$ ,  $a \in \mathbb{C}\mathcal{C}\ell(n)$ , sends  $\Psi(n)$  onto  $\Psi'(n)$ .  $\square$

Given a spinor space  $\Psi(n)$ , let  $G\Psi(n)$  be a subgroup of  $PGL(2^{n/2}, \mathbb{C})$  which preserves  $\Psi(n)$ . Then it follows from Lemma 4.7 that a set of spinor spaces  $S\Psi(n)$  is bijective to the quotient

$$S\Psi(n) = PGL(2^{n/2}, \mathbb{C})/G\Psi(n). \quad (4.2)$$

For instance, let  $\Psi_{11}(n)$  be a spinor space in Example 4.1. Its stabilizer  $G\Psi_{11}(n)$  consists of inner automorphisms generated by elements  $g \in \text{Mat}(2^{n/2}, \mathbb{C})$  with components  $g_{k1} = 0$ ,  $1 < k$ .

## 5 Reduced structures

This section addresses gauge theory on principal bundles in a case of spontaneous symmetry breaking [4, 10, 19].

### 5.1 Reduced structures in gauge theory

Let  $G$  be a real Lie group whose unit is denoted by  $\mathbf{1}$ . A fibre bundle

$$\pi_P : P \rightarrow X \quad (5.1)$$

is called a principal bundle with a structure group  $G$  if it admits an action of  $G$  on  $P$  on the right by a fibrewise morphism

$$R_P : G \times_X P \xrightarrow{X} P, \quad R_g : p \rightarrow pg, \quad \pi_P(p) = \pi_P(pg), \quad p \in P, \quad (5.2)$$

which is free and transitive on each fibre of  $P$ . It follows that:

- a typical fibre of  $P$  (5.1) is a group space of  $G$ , which a structure group  $G$  acts on by left multiplications;
- the quotient of  $P$  with respect to the action (5.2) of  $G$  is diffeomorphic to a base  $X$ , i.e.,  $P/G = X$ ;
- a principal bundle  $P$  is equipped with a bundle atlas

$$\Psi_P = \{(U_\alpha, \psi_\alpha^P), \varrho_{\alpha\beta}\} \quad (5.3)$$

whose trivialization morphisms

$$\psi_\alpha^P : \pi_P^{-1}(U_\alpha) \rightarrow U_\alpha \times G$$

obey a condition

$$\psi_\alpha^P(pg) = g\psi_\alpha^P(p), \quad g \in G, \quad (5.4)$$

and transition functions  $\varrho_{\alpha\beta}$  are local  $G$ -valued functions.

For short, we call  $P$  (5.1) the principal  $G$ -bundle.

Due to the property (5.4), every trivialization morphism  $\psi_\alpha^P$  determines a unique local section  $z_\alpha : U_\alpha \rightarrow P$  such that

$$(\psi_\alpha^P \circ z_\alpha)(x) = \mathbf{1}, \quad x \in U_\alpha.$$

A transformation law for  $z_\alpha$  reads

$$z_\beta(x) = z_\alpha(x)\varrho_{\alpha\beta}(x), \quad x \in U_\alpha \cap U_\beta. \quad (5.5)$$

Conversely, a family

$$\Psi_P = \{(U_\alpha, z_\alpha), \varrho_{\alpha\beta}\} \quad (5.6)$$

of local sections of  $P$  which obey the transformation law (5.5) determines the unique bundle atlas  $\Psi_P$  (5.3) of a principal bundle  $P$ .

**COROLLARY 5.1:** It follows that a principal bundle admits a global section iff it is trivial.  $\square$

**Example 5.1:** Let  $H$  be a closed subgroup of a real Lie group  $G$ . Then  $H$  is a Lie group. Let  $G/H$  be the quotient of  $G$  with respect to an action of  $H$  on  $G$  by right multiplications. Then

$$\pi_{GH} : G \rightarrow G/H \quad (5.7)$$

is a principal  $H$ -bundle. If  $H$  is a maximal compact subgroup of  $G$ , the quotient  $G/H$  is diffeomorphic to an Euclidean manifold  $\mathbb{R}^m$  and the principal bundle (5.7) is trivial, i.e.,  $G$  is diffeomorphic to a product  $\mathbb{R}^m \times H$ .  $\square$

**Remark 5.2:** If  $f : X' \rightarrow X$  is a manifold morphism, the pull-back  $f^*P \rightarrow X'$  of a principal bundle also is a principal bundle with the same structure group as of  $P$ .  $\square$

**Remark 5.3:** Let  $P \rightarrow X$  and  $P' \rightarrow X'$  be principal  $G$ - and  $G'$ -bundles, respectively. A bundle morphism  $\Phi : P \rightarrow P'$  is a morphism of principal bundles if there exists a Lie group homomorphism  $\gamma : G \rightarrow G'$  such that  $\Phi(pg) = \Phi(p)\gamma(g)$ .  $\square$

In accordance with Remark 5.3, an automorphism  $\Phi_P$  of a principal  $G$ -bundle  $P$  is called principal if it is equivariant under the right action (5.2) of a structure group  $G$  on  $P$ , i.e.,

$$\Phi_P(pg) = \Phi_P(p)g, \quad g \in G, \quad p \in P. \quad (5.8)$$

In particular, every vertical principal automorphism of a principal bundle  $P$  is represented as

$$\Phi_P(p) = pf(p), \quad p \in P, \quad (5.9)$$

where  $f$  is a  $G$ -valued equivariant function on  $P$ , i.e.,

$$f(pg) = g^{-1}f(p)g, \quad g \in G. \quad (5.10)$$

Note that there is one-to-one correspondence

$$s(\pi_P(p))p = pf(p), \quad p \in P, \quad (5.11)$$

between the equivariant functions  $f$  (5.10) (consequently, the vertical automorphisms of  $P$ ) and the global sections  $s$  of the group bundle  $P^G$  (5.13) (Example 5.4).

Let  $P$  (5.1) be a principal bundle and  $V$  a smooth manifold that on a group  $G$  acts on the left. Let us consider the quotient

$$Y = (P \times V)/G \quad (5.12)$$

of a product  $P \times V$  by identification of elements  $(p, v)$  and  $(pg, g^{-1}v)$  for all  $g \in G$ . It is a fibre bundle with a structure group  $G$  and a typical fibre  $V$  which



is said to be associated to the principal  $G$ -bundle  $P$ . For the sake of brevity, we call it the  $P$ -associated bundle.

**Example 5.4:** A  $P$ -associated group bundle is defined as the quotient

$$\pi_{P^G} : P^G = (P \times G)/G \rightarrow X, \quad (5.13)$$

where a structure group  $G$  which acts on itself by the adjoint representation. There is the following fibre-to-fibre action of the group bundle  $P^G$  on any  $P$ -associated bundle  $Y$  (5.12):

$$P^G \times_X Y \rightarrow_X Y, \quad ((p, g)/G, (p, v)/G) \rightarrow (p, gv)/G, \quad g \in G, \quad v \in V.$$

For instance, the action of  $P^G$  on  $P$  in the formula (5.11) is of this type.  $\square$

The peculiarity of the  $P$ -associated bundle  $Y$  (5.12) is the following.

- Every bundle atlas  $\Psi_P = \{(U_\alpha, z_\alpha)\}$  (5.6) of  $P$  defines a unique associated bundle atlas

$$\Psi = \{(U_\alpha, \psi_\alpha(x) = [z_\alpha(x)]^{-1})\} \quad (5.14)$$

of the quotient  $Y$  (5.12).

- Any principal automorphism  $\Phi_P$  (5.8) of  $P$  yields a unique principal automorphism

$$\Phi_Y : (p, v)/G \rightarrow (\Phi_P(p), v)/G, \quad p \in P, \quad v \in V, \quad (5.15)$$

of the  $P$ -associated bundle  $Y$  (5.12).

**Remark 5.5:** In classical gauge theory on a principal bundle  $P$ , matter fields are described as sections of  $P$ -associated bundles (5.12).  $\square$

As was mentioned above, spontaneous symmetry breaking in classical gauge theory on a principal bundle  $P \rightarrow X$  is characterized by a reduction of a structure group of  $P$  [4, 16, 22]

Let  $H$  and  $G$  be Lie groups and  $\phi : H \rightarrow G$  a Lie group homomorphism. If  $P_H \rightarrow X$  is a principal  $H$ -bundle, there always exists a principal  $G$ -bundle  $P_G \rightarrow X$  together with the principal bundle morphism

$$\Phi : P_H \xrightarrow[X]{} P_G \quad (5.16)$$

over  $X$  (Remark 5.3). It is a  $P_H$ -associated bundle

$$P_G = (P_H \times G)/H$$

with a typical fibre  $G$  which on  $H$  acts on the left by the rule  $h(g) = \phi(h)g$ , while  $G$  acts on  $P_G$  as

$$G \ni g' : (p, g)/H \rightarrow (p, gg')/H.$$

Conversely, if  $P_G \rightarrow X$  is a principal  $G$ -bundle, a problem is to find a principal  $H$ -bundle  $P_H \rightarrow X$  together with the principal bundle morphism

(5.16). If  $H \rightarrow G$  is a group epimorphism, one says that  $P_G$  gives rise to  $P_H$ . If  $H \rightarrow G$  is a closed subgroup, we have the structure group reduction. In this case, the bundle monomorphism (5.16) is called a reduced  $H$ -structure.

Let  $P$  (5.1) be a principal  $G$ -bundle, and let  $H$ ,  $\dim H > 0$ , be a closed subgroup of  $G$ . Then we have a composite bundle

$$P \rightarrow P/H \rightarrow X, \quad (5.17)$$

where

$$P_\Sigma = P \xrightarrow{\pi_{P\Sigma}} P/H \quad (5.18)$$

is a principal bundle with a structure group  $H$  and

$$\Sigma = P/H \xrightarrow{\pi_{\Sigma X}} X \quad (5.19)$$

is a  $P$ -associated bundle with a typical fibre  $G/H$  which on a structure group  $G$  acts on the left (Example 5.1).

DEFINITION 5.1: One says that a structure Lie group  $G$  of a principal bundle  $P$  is reduced to its closed subgroup  $H$  if the following equivalent conditions hold.

- A principal bundle  $P$  admits a bundle atlas  $\Psi_P$  (5.3) with  $H$ -valued transition functions  $\varrho_{\alpha\beta}$ .
- There exists a reduced principal subbundle  $P_H$  of  $P$  with a structure group  $H$ .  $\square$

**Remark 5.6:** It is easily justified that these conditions are equivalent. If  $P_H \subset P$  is a reduced principal subbundle, its atlas (5.6) given by local sections  $z_\alpha$  of  $P_H \rightarrow X$  is a desired atlas of  $P$ . Conversely, let  $\Psi_P = \{(U_\alpha, z_\alpha), \varrho_{\alpha\beta}\}$  (5.6) be an atlas of  $P$  with  $H$ -valued transition functions  $\varrho_{\alpha\beta}$ . For any  $x \in U_\alpha \subset X$ , let us define a submanifold  $z_\alpha(x)H \subset P_x$ . These submanifolds form a desired  $H$ -subbundle of  $P$  because

$$z_\alpha(x)H = z_\beta(x)H \varrho_{\beta\alpha}(x)$$

on the overlaps  $U_\alpha \cap U_\beta$ .  $\square$

A key point is the following.

THEOREM 5.2: There is one-to-one correspondence

$$P^h = \pi_{P\Sigma}^{-1}(h(X)) \quad (5.20)$$

between the reduced principal  $H$ -subbundles  $i_h : P^h \rightarrow P$  of  $P$  and the global sections  $h$  of the quotient bundle  $P/H \rightarrow X$  (5.19) [4, 7].  $\square$

In classical field theory, global sections of a quotient bundle  $P/H \rightarrow X$  are interpreted as classical Higgs fields [4, 12, 16, 22].

COROLLARY 5.3: A glance at the formula (5.20) shows that a reduced principal  $H$ -bundle  $P^h$  is the restriction  $h^*P_\Sigma$  of a principal  $H$ -bundle  $P_\Sigma$  (5.18) to  $h(X) \subset \Sigma$ . Any atlas  $\Psi^h$  of a principal  $H$ -bundle  $P^h$  defined by a family of

local sections of  $P^h \rightarrow X$  also is an atlas of a principal  $G$ -bundle  $P$  and the  $P$ -associated bundle  $\Sigma \rightarrow X$  (5.19) with  $H$ -valued transition functions (Remark 5.6). Herewith, a Higgs field  $h$  written with respect to an atlas  $\Psi^h$  takes its values into the center of a quotient  $G/H$ .  $\square$

**Remark 5.7:** Let  $P^h$  be a reduced principal  $H$ -subbundle of a principal  $G$ -bundle in Corollary 5.3. Any principal automorphism  $g\phi$  of  $P^h$  gives rise to a principal automorphism of  $P$  by means of the relation  $\phi(P^h g) = \phi(P^h)g$ ,  $g \in G$ .  $\square$

In general, there is topological obstruction to reduction of a structure group of a principal bundle to its subgroup.

**THEOREM 5.4:** The structure group  $G$  of a principal bundle  $P$  always is reducible to its closed subgroup  $H$ , if the quotient  $G/H$  is diffeomorphic to a Euclidean space  $\mathbb{R}^m$ .  $\square$

In particular, this is the case of a maximal compact subgroup  $H$  of a Lie group  $G$  (Example 5.1). Then the following is a corollary of Theorem 5.4 [24].

**THEOREM 5.5:** A structure group  $G$  of a principal bundle always is reducible to its maximal compact subgroup  $H$ .  $\square$

Given different Higgs fields  $h$  and  $h'$ , the corresponding principal  $H$ -subbundles  $P^h$  and  $P^{h'}$  of a principal  $G$ -bundle  $P$  fail to be isomorphic to each other in general [4, 16].

**THEOREM 5.6:** Let a structure Lie group  $G$  of a principal bundle be reducible to its closed subgroup  $H$ .

- Every vertical principal automorphism  $\Phi$  of  $P$  sends a reduced principal  $H$ -subbundle  $P^h$  of  $P$  onto an isomorphic principal  $H$ -subbundle  $P^{h'}$ .
- Conversely, let two reduced principal subbundles  $P^h$  and  $P^{h'}$  of a principal bundle  $P \rightarrow X$  be isomorphic to each other, and let  $\Phi : P^h \rightarrow P^{h'}$  be their isomorphism over  $X$ . Then  $\Phi$  is extended to a vertical principal automorphism of  $P$ .  $\square$

*Proof:* Let

$$\Psi^h = \{(U_\alpha, z_\alpha^h), \varrho_{\alpha\beta}^h\} \quad (5.21)$$

be an atlas of a reduced principal subbundle  $P^h$ , where  $z_\alpha^h$  are local sections of  $P^h \rightarrow X$  and  $\varrho_{\alpha\beta}^h$  are the transition functions. Given a vertical automorphism  $\Phi$  of  $P$ , let us provide a subbundle  $P^{h'} = \Phi(P^h)$  with an atlas

$$\Psi^{h'} = \{(U_\alpha, z_\alpha^{h'}), \varrho_{\alpha\beta}^{h'}\} \quad (5.22)$$

given by the local sections  $z_\alpha^{h'} = \Phi \circ z_\alpha^h$  of  $P^{h'} \rightarrow X$ . Then it is readily observed that

$$\varrho_{\alpha\beta}^{h'}(x) = \varrho_{\alpha\beta}^h(x), \quad x \in U_\alpha \cap U_\beta. \quad (5.23)$$

Conversely, any isomorphism  $(\Phi, \text{Id } X)$  of reduced principal subbundles  $P^h$  and  $P^{h'}$  of  $P$  defines an  $H$ -equivariant  $G$ -valued function  $f$  on  $P^h$  given by the relation

$$pf(p) = \Phi(p), \quad p \in P^h.$$

Its prolongation to a  $G$ -equivariant function on  $P$  is defined as

$$f(pg) = g^{-1}f(p)g, \quad p \in P^h, \quad g \in G.$$

In accordance with the relation (5.9), this function provides a vertical principal automorphism of  $P$  whose restriction to  $P^h$  coincides with  $\Phi$ .  $\square$

**THEOREM 5.7:** If the quotient  $G/H$  is homeomorphic to a Euclidean space  $\mathbb{R}^m$ , all principal  $H$ -subbundles of a principal  $G$ -bundle  $P$  are isomorphic to each other [24].  $\square$

**Remark 5.8:** Let  $P^h$  and  $P^{h'}$  be isomorphic reduced principal subbundles in Theorem 5.6. A principal  $G$ -bundle  $P$  provided with the atlas  $\Psi^h$  (5.21) can be regarded as a  $P^h$ -associated bundle with a structure group  $H$  acting on its typical fibre  $G$  on the left. Endowed with the atlas  $\Psi^{h'}$  (5.22), it is a  $P^{h'}$ -associated  $H$ -bundle. The  $H$ -bundles  $(P, \Psi^h)$  and  $(P, \Psi^{h'})$  fail to be equivalent because their atlases  $\Psi^h$  and  $\Psi^{h'}$  are not equivalent. Indeed, the union of these atlases is an atlas

$$\Psi = \{(U_\alpha, z_\alpha^h, z_\alpha^{h'}), \varrho_{\alpha\beta}^h, \varrho_{\alpha\beta}^{h'}, \varrho_{\alpha\alpha} = f(z_\alpha)\}$$

possessing transition functions

$$z_\alpha^{h'} = z_\alpha^h \varrho_{\alpha\alpha}, \quad \varrho_{\alpha\alpha}(x) = f(z_\alpha(x)), \quad (5.24)$$

between the bundle charts  $(U_\alpha, z_\alpha^h)$  and  $(U_\alpha, z_\alpha^{h'})$  of  $\Psi^h$  and  $\Psi^{h'}$ , respectively. However, the transition functions  $\varrho_{\alpha\alpha}$  are not  $H$ -valued. At the same time, a glance at the equalities (5.23) shows that transition functions of both the atlases form the same cocycle. Consequently, the  $H$ -bundles  $(P, \Psi^h)$  and  $(P, \Psi^{h'})$  are associated. Due to the isomorphism  $\Phi : P^h \rightarrow P^{h'}$ , one can write

$$\begin{aligned} P &= (P^h \times G)/H = (P^{h'} \times G)/H, \\ (p \times g)/H &= (\Phi(p) \times f^{-1}(p)g)/H. \end{aligned}$$

For any  $\rho \in H$ , we have

$$\begin{aligned} (p\rho, g)/H &= (\Phi(p)\rho, f^{-1}(p)g)/H = (\Phi(p), \rho f^{-1}(p)g)/H = \\ &= (\Phi(p), f^{-1}(p)\rho'g)/H, \end{aligned}$$

where

$$\rho' = f(p)\rho f^{-1}(p). \quad (5.25)$$

It follows that  $(P, \Psi^{h'})$  can be regarded as a  $P^h$ -associated bundle with the same typical fibre  $G$  as that of  $(P, \Psi^h)$ , but the action  $g \rightarrow \rho'g$  (5.25) of a structure group  $H$  on a typical fibre of  $(P, \Psi^{h'})$  is not equivalent to its action  $g \rightarrow \rho g$  on a typical fibre of  $(P, \Psi^h)$  since they possess different orbits in  $G$ .  $\square$

Given a classical Higgs field  $h$  and the corresponding reduced principal  $H$ -bundle  $P^h$ , let

$$Y^h = (P^h \times V)/H \quad (5.26)$$

be the associated vector bundle with a typical fibre  $V$  which admits a representation of a group  $H$  of exact symmetries. Its sections  $s_h$  describe matter fields in the presence of a classical Higgs field  $h$  (Remark 5.5).

In general, the fibre bundle  $Y^h$  (5.26) fails to be associated to another principal  $H$ -subbundles  $P^{h'}$  of  $P$ . It follows that, in this case, a  $V$ -valued matter field can be represented only by a pair with a certain Higgs field. Therefore, a goal is to describe the totality of these pairs  $(s_h, h)$  for all Higgs fields  $h \in \Sigma(X)$ .

**Remark 5.9:** If reduced principal  $H$ -subbundles  $P^h$  and  $P^{h'}$  of a principal  $G$ -bundle are isomorphic in accordance with Theorem 5.6, then the  $P^h$ -associated bundle  $Y^h$  (5.26) is associated as

$$Y^h = (\Phi(p) \times V)/H \quad (5.27)$$

to  $P^{h'}$ . If a typical fibre  $V$  admits an action of the whole group  $G$ , the  $P^h$ -associated bundle  $Y^h$  (5.26) also is  $P$ -associated as

$$Y^h = (P^h \times V)/H = (P \times V)/G.$$

Such  $P$ -associated bundles  $P^h$  and  $P^{h'}$  are equivalent as  $G$ -bundles, but they fail to be equivalent as  $H$ -bundles because transition functions between their atlases are not  $H$ -valued (Remark 5.8).  $\square$

In order to describe matter fields in the presence of different classical Higgs fields, let us consider the composite bundle (5.17) and the composite bundle

$$Y \xrightarrow{\pi_{Y\Sigma}} \Sigma \xrightarrow{\pi_{\Sigma X}} X \quad (5.28)$$

where  $Y \rightarrow \Sigma$  is a  $P_\Sigma$ -associated bundle

$$Y = (P \times V)/H \quad (5.29)$$

with a structure group  $H$ . Given a Higgs field  $h$  and the corresponding reduced principal  $H$ -subbundle  $P^h = h^*P$ , the  $P^h$ -associated fibre bundle (5.26) is the restriction

$$Y^h = h^*Y = (h^*P \times V)/H \quad (5.30)$$

of a fibre bundle  $Y \rightarrow \Sigma$  to  $h(X) \subset \Sigma$ . Every global section  $s_h$  of the fibre bundle  $Y^h$  (5.30) is a global section of the composite bundle (5.28) projected onto a Higgs field  $h = \pi_{Y\Sigma} \circ s$ . Conversely, every global section  $s$  of the composite bundle  $Y \rightarrow X$  (5.28) projected onto a Higgs field  $h = \pi_{Y\Sigma} \circ s$  takes its

values into the subbundle  $Y^h Y$  (5.30). Hence, there is one-to-one correspondence between the sections of the fibre bundle  $Y^h$  (5.26) and the sections of the composite bundle (5.28) which cover  $h$ .

Thus, it is the composite bundle  $Y \rightarrow X$  (5.28) whose sections describe the above mentioned totality of pairs  $(s_h, h)$  of matter fields and Higgs fields in classical gauge theory with spontaneous symmetry breaking [4, 16, 22].

A key point is that, though  $Y \rightarrow \Sigma$  is a fibre bundle with a structure group  $H$ , a composite bundle  $Y \rightarrow X$  is a  $P$ -associated bundle as follows [4, 22].

**THEOREM 5.8:** The composite bundle  $Y \rightarrow X$  (5.28) is a  $P$ -associated bundle

$$\begin{aligned} Y &= (P \times (G \times V)/H)/G, \\ (pg', (g\rho, v)) &= (pg', (g, \rho v)) = (p, g'(g, \rho v)) = (p, (g'g, \rho v)). \end{aligned}$$

with a structure group  $G$ . Its typical fibre is a fibre bundle

$$\pi_{WH} : W = (G \times V)/H \rightarrow G/H \quad (5.31)$$

associated to a principal  $H$ -bundle  $G \rightarrow G/H$  (5.7). A structure group  $G$  acts on  $W$  by the law

$$g' : (G \times V)/H \rightarrow (g'G \times V)/H. \quad (5.32)$$

□

**THEOREM 5.9:** Given a Higgs field  $h$ , any atlas of a  $P_\Sigma$ -associated bundle  $Y \rightarrow \Sigma$  defines an atlas of a  $P$ -associated bundle  $Y \rightarrow X$  with  $H$ -valued transition functions. The converse need not be true. □

*Proof:* Any atlas  $\Psi_{Y\Sigma}$  of a  $P_\Sigma$ -associated bundle  $Y \rightarrow \Sigma$  is defined by an atlas

$$\Psi_{P\Sigma} = \{(U_{\Sigma\iota}, z_\iota), \varrho_{\iota\kappa}\} \quad (5.33)$$

of the principal  $H$ -bundle  $P_\Sigma$  (5.18). Given a section  $h$  of  $\Sigma \rightarrow X$ , we have an atlas

$$\Psi^h = \{(\pi_{P\Sigma}(U_{\Sigma\iota}), z_\iota \circ h), \varrho_{\iota\kappa} \circ h\} \quad (5.34)$$

of the reduced principal  $H$ -bundle  $P^h$  which also is an atlas of  $P$  with  $H$ -valued transition functions (Remark 5.6). □

Given an atlas  $\Psi_P$  of  $P$ , the quotient bundle  $\Sigma \rightarrow X$  (5.19) is endowed with the associated atlas (5.14). With this atlas and an atlas  $\Psi_{Y\Sigma}$  of  $Y \rightarrow \Sigma$ , the composite bundle  $Y \rightarrow X$  (5.28) is provided with adapted bundle coordinates  $(x^\lambda, \sigma^m, y^i)$  where  $(\sigma^m)$  are fibre coordinates on  $\Sigma \rightarrow X$  and  $(y^i)$  are those on  $Y \rightarrow \Sigma$ .

**THEOREM 5.10:** Any principal automorphism of a principal  $G$ -bundle  $P \rightarrow X$  is  $G$ -equivariant and, consequently,  $H$ -equivariant. Thus, it is a principal automorphism of a principal  $H$ -bundle  $P \rightarrow \Sigma$  and, consequently, it yields an automorphism of the  $P_\Sigma$ -associated bundle  $Y$  (5.28). □

The converse is not true. For instance, a vertical principal automorphism of  $P \rightarrow \Sigma$  is never a principal automorphism of  $P \rightarrow X$ .

Theorems 5.8 – 5.10 enables one to describe matter fields with an exact symmetry group  $H \subset G$  in the framework of gauge theory on a  $G$ -principal bundle  $P \rightarrow X$  if its structure group  $G$  is reducible to  $H$ .

## 5.2 Lorentz reduced structures in gravitation theory

As was mentioned in Section 1, gravitation theory based on Relativity and Equivalence Principles is formulated as gauge theory on natural bundles (Remark 1.1) over a world manifold whose structure group  $GL_4$  (1.1) is reduced to a Lorentz subgroup  $SO(1, 3)$  [4, 14, 18].

Natural bundles are exemplified by tensor bundles  $T$  and, in particular, the tangent bundle  $TX$  over  $X$ . Given a diffeomorphism  $f$  of  $X$ , the tangent morphism  $Tf : TX \rightarrow TX$  is a general covariant transformation of  $TX$ . Tensor bundles over an oriented world manifold possess the structure group  $GL_4$  (1.1). An associated principal bundle is the above mentioned frame bundle  $LX$  (Remark 1.1). Its (local) sections are called frame fields. Given the holonomic atlas of the tangent bundle  $TX$ , every element  $\{H_a\}$  of a frame bundle  $LX$  takes a form  $H_a = H_a^\mu \partial_\mu$ , where  $H_a^\mu$  is a matrix of the natural representation of a group  $GL_4$  in  $\mathbb{R}^4$ . These matrices constitute bundle coordinates

$$(x^\lambda, H_a^\mu), \quad H_a'^\mu = \frac{\partial x'^\mu}{\partial x^\lambda} H_a^\lambda,$$

on  $LX$  associated to its holonomic atlas

$$\Psi_T = \{(U_\iota, z_\iota = \{\partial_\mu\})\}, \quad (5.35)$$

given by local frame fields  $z_\iota = \{\partial_\mu\}$ . With respect to these coordinates, the canonical right action of  $GL_4$  on  $LX$  reads  $GL_4 \ni g : H_a^\mu \rightarrow H_b^\mu g^b_a$ .

A frame bundle  $LX$  is equipped with a canonical  $\mathbb{R}^4$ -valued one-form

$$\theta_{LX} = H_\mu^a dx^\mu \otimes t_a, \quad (5.36)$$

where  $\{t_a\}$  is a fixed basis for  $\mathbb{R}^4$  and  $H_\mu^a$  is the inverse matrix of  $H_a^\mu$ .

A frame bundle  $LX \rightarrow X$  is natural. Any diffeomorphism  $f$  of  $X$  gives rise to a principal automorphism

$$\tilde{f} : (x^\lambda, H_a^\lambda) \rightarrow (f^\lambda(x), \partial_\mu f^\lambda H_a^\mu) \quad (5.37)$$

of  $LX$  which is its general covariant transformation.

Let  $Y = (LX \times V)/GL_4$  be an  $LX$ -associated bundle with a typical fibre  $V$ . It admits a lift of any diffeomorphism  $f$  of its base to an automorphism

$$f_Y(Y) = (\tilde{f}(LX) \times V)/GL_4$$

of  $Y$  associated with the principal automorphism  $\tilde{f}$  (5.37) of a frame bundle  $LX$ . Thus, all bundles associated to a frame bundle  $LX$  are natural bundles.

As was mentioned above, gravitation theory on a world manifold  $X$  is a gauge theory with spontaneous symmetry breaking described by Lorentz reduced structures of a frame bundle  $LX$ . We deal with the following Lorentz and proper Lorentz reduced structures.

By a Lorentz reduced structure is meant a reduced  $SO(1, 3)$ -subbundle  $L^g X$ , called the Lorentz subbundle, of a frame bundle  $LX$ .

Let  $L = SO^0(1, 3)$  be a proper Lorentz group. Recall that  $SO(1, 3) = \mathbb{Z}_2 \times L$ . A proper Lorentz reduced structure is defined as a reduced  $L$ -subbundle  $L^h X$  of  $LX$ .

**THEOREM 5.11:** If a world manifold  $X$  is simply connected, there is one-to-one correspondence between the Lorentz and proper Lorentz reduced structures.  $\square$

One can show that different proper Lorentz subbundles  $L^h X$  and  $L^{h'} X$  of a frame bundle  $LX$  are isomorphic as principal  $L$ -bundles. This means that there exists a vertical automorphism of a frame bundle  $LX$  which sends  $L^h X$  onto  $L^{h'} X$  [4, 16]. If a world manifold  $X$  is simply connected, the similar property of Lorentz subbundles also is true in accordance with Theorem 5.11.

**Remark 5.10:** There is the well-known topological obstruction to the existence of a Lorentz structure on a world manifold  $X$ . All non-compact manifolds and compact manifolds whose Euler characteristic equals zero admit a Lorentz reduced structure [3].  $\square$

By virtue of Theorem 5.2, there is one-to-one correspondence between the principal  $L$ -subbundles  $L^h X$  of a frame bundle  $LX$  and the global sections  $h$  of a quotient bundle

$$\Sigma_T = LX/L, \quad (5.38)$$

called the tetrad bundle. This is an  $LX$ -associated bundle with the typical fibre  $GL_4/L$ . Its global sections are called the tetrad fields. The fibre bundle (5.38) is a two-fold covering  $\zeta : \Sigma_T \rightarrow \Sigma_{PR}$  of the quotient bundle

$$\Sigma_{PR} = LX/SO(1, 3). \quad (5.39)$$

whose global sections  $g$  are pseudo-Riemannian metrics of signature  $(+, - - -)$  on a world manifold/ It is called the metric bundle.

In particular, every tetrad field  $h$  defines a unique pseudo-Riemannian metric  $g = \zeta \circ h$ . For the sake of convenience, one usually identifies a metric bundle with an open subbundle of the tensor bundle  $\Sigma_{PR} \subset \overset{2}{V}TX$ . Therefore, the metric bundle  $\Sigma_{PR}$  (5.39) can be equipped with bundle coordinates  $(x^\lambda, \sigma^{\mu\nu})$ .

Every tetrad field  $h$  defines an associated Lorentz bundle atlas

$$\Psi^h = \{(U_\iota, z_\iota^h = \{h_a\})\} \quad (5.40)$$

of a frame bundle  $LX$  such that the corresponding local sections  $z_\iota^h$  of  $LX$  take their values into a proper Lorentz subbundle  $L^h X$  and the transition functions of  $\Psi^h$  (5.40) between the frames  $\{h_a\}$  are  $L$ -valued. The frames (5.40):

$$\{h_a = h_a^\mu(x)\partial_\mu\}, \quad h_a^\mu = H_a^\mu \circ z_\iota^h, \quad x \in U_\iota, \quad (5.41)$$



are called the tetrad frames.

Given a Lorentz bundle atlas  $\Psi^h$ , the pull-back

$$h = h^a \otimes t_a = z_\iota^{h*} \theta_{LX} = h_\lambda^a(x) dx^\lambda \otimes t_a \quad (5.42)$$

of the canonical form  $\theta_{LX}$  (5.36) by a local section  $z_\iota^h$  is called the (local) tetrad form. It determines tetrad coframes

$$\{h^a = h_\mu^a(x) dx^\mu\}, \quad x \in U_\iota, \quad (5.43)$$

in the cotangent bundle  $T^*X$ . They are the dual of the tetrad frames (5.41). The coefficients  $h_a^\mu$  and  $h_\mu^a$  of the tetrad frames (5.41) and coframes (5.43) are called the tetrad functions. They are transition functions between the holonomic atlas  $\Psi_T$  (5.35) and the Lorentz atlas  $\Psi^h$  (5.40) of a frame bundle  $LX$ .

With respect to the Lorentz atlas  $\Psi^h$  (5.40), a tetrad field  $h$  can be represented by the  $\mathbb{R}^4$ -valued tetrad form (5.42). Relative to this atlas, the corresponding pseudo-Riemannian metric  $g = \zeta \circ h$  takes the well-known form

$$g = \eta(h \otimes h) = \eta_{ab} h^a \otimes h^b, \quad g_{\mu\nu} = h_\mu^a h_\nu^b \eta_{ab}, \quad (5.44)$$

where  $\eta = \text{diag}(1, -1, -1, -1)$  is the Minkowski metric in  $\mathbb{R}^4$  written with respect to its fixed basis  $\{t_a\}$ . It is readily observed that the tetrad coframes  $\{h^a\}$  (5.43) and the tetrad frames  $\{h_a\}$  (5.41) are orthonormal relative to the pseudo-Riemannian metric (5.44), namely:

$$g^{\mu\nu} h_\mu^a h_\nu^b = \eta^{ab}, \quad g_{\mu\nu} h_a^\mu h_b^\nu = \eta_{ab}.$$

## 6 Spinor structures

As was mentioned above, we aim to describe spinor bundles as subbundles of a fibre bundle in complex Clifford algebras.

### 6.1 Fibre bundles in Clifford algebras

One usually consider fibre bundles in Clifford algebras whose structure group is a group of automorphisms of these algebras [4, 9] (Remark 6.1). A problem is that this group fails to preserve spinor subspaces of a Clifford algebra and, thus, it can not be a structure group of spinor bundles. Therefore, we define fibre bundles in Clifford algebras whose structure group is a group of invertible elements of a Clifford algebra which acts on this algebra by left multiplications. Certainly, it preserves minimal left ideals of this algebra and, consequently, it is a structure group of spinor bundles. In a case in question, this is a matrix group.

Let  $\mathbb{C}\mathcal{C}\ell(n)$  be a complex Clifford algebra modelled over an even dimensional complex space  $\mathbb{C}^n$  (Definition 2.4). It is isomorphic to a ring  $\text{Mat}(2^{n/2}, \mathbb{C})$  of complex  $(2^{n/2} \times 2^{n/2})$ -matrices (Theorem 2.9). Its invertible elements constitute a general linear group  $GL(2^{n/2}, \mathbb{C})$  whose adjoint representation in  $\mathbb{C}\mathcal{C}\ell(n)$  yields

the projective linear group  $PGL(2^{n/2}, \mathbb{C})$  (3.31) of automorphisms of  $\mathbb{C}\ell(n)$  (Theorem 3.7).

DEFINITION 6.1: Given a smooth manifold  $X$ , let us consider a principal bundle  $P \rightarrow X$  with a structure group  $GL(2^{n/2}, \mathbb{C})$ . A fibre bundle in complex Clifford algebras  $\mathbb{C}\ell(n)$  is defined to be the  $P$ -associated bundle (5.12):

$$\mathcal{C} = (P \times \text{Mat}(2^{n/2}, \mathbb{C})) / GL(2^{n/2}, \mathbb{C}) \rightarrow X \quad (6.1)$$

with a typical fibre

$$\mathbb{C}\ell(n) = \text{Mat}(2^{n/2}, \mathbb{C}) \quad (6.2)$$

which carries out the left-regular representation of a group  $GL(2^{n/2}, \mathbb{C})$ .  $\square$

Owing to the canonical inclusion  $GL(2^{n/2}, \mathbb{C}) \rightarrow \text{Mat}(2^{n/2}, \mathbb{C})$ , a principal  $GL(2^{n/2}, \mathbb{C})$ -bundle  $P$  is a subbundle  $P \subset \mathcal{C}$  of the Clifford algebra bundle  $\mathcal{C}$  (6.1). Herewith, the canonical right action of a structure group  $GL(2^{n/2}, \mathbb{C})$  on a principal bundle  $P$  is extended to the fibrewise action of  $GL(2^{n/2}, \mathbb{C})$  on the Clifford algebra bundle  $\mathcal{C}$  (6.1) by right multiplications. This action is globally defined because it is commutative with transition functions of  $\mathcal{C}$  acting on its typical fibre  $\text{Mat}(2^{n/2}, \mathbb{C})$  on the left.

**Remark 6.1:** As was mentioned above, one usually considers a fibre bundle in Clifford algebras  $\text{Mat}(2^{n/2})$  whose structure group is the projective linear group  $PGL(2^{n/2}, \mathbb{C})$  (3.31) of automorphisms of  $\mathbb{C}\ell(n)$ . This also is a  $P$ -associated bundle

$$\mathcal{AC} = (P \times \mathbb{C}\ell(n)) / GL(2^{n/2}, \mathbb{C}) \rightarrow X \quad (6.3)$$

where  $GL(2^{n/2}, \mathbb{C})$  acts on  $\mathbb{C}\ell(n)$  by the adjoint representation. In particular, a certain subbundle of  $\mathcal{AC}$  (6.3) is the group bundle  $P^G$  (5.13) (Remark 5.4).  $\square$

Let  $\Psi(n)$  (Definition 4.2) be a spinor space of a complex Clifford algebra  $\mathbb{C}\ell(n)$ . Being a minimal left ideal of  $\mathbb{C}\ell(n)$  (Definition 4.3), it is a subspace  $\Psi(n)$  of  $\mathbb{C}\ell(n)$  (Theorem 4.1) which inherits the left-regular representation of a group  $GL(2^{n/2}, \mathbb{C})$  in  $\mathbb{C}\ell(n)$ .

DEFINITION 6.2: Given a principal  $GL(2^{n/2}, \mathbb{C})$ -bundle  $P$ , a spinor bundle is defined as a  $P$ -associated bundle

$$S = (P \times \Psi(n)) / GL(2^{n/2}, \mathbb{C}) \rightarrow X \quad (6.4)$$

with a typical fibre  $\Psi(n) = \mathbb{C}^{2^{n/2}}$  and a structure group  $GL(2^{n/2}, \mathbb{C})$  which acts on  $\Psi(n)$  by left multiplications that is equivalent to the natural matrix representation of  $GL(2^{n/2}, \mathbb{C})$  in  $\mathbb{C}^{2^{n/2}}$  (Corollary 4.2).  $\square$

Obviously, the spinor bundle  $S$  (6.4) is a subbundle of the Clifford algebra bundle  $\mathcal{C}$  (6.1). However,  $S$  (6.4) need not be a subbundle of the fibre bundle  $\mathcal{AC}$  (6.3) in Clifford algebras because a spinor space  $\Psi(n)$  is not stable under automorphisms of a Clifford algebra  $\mathbb{C}\ell(n)$ .

At the same time, given the spinor representation (1.2), of a complex Clifford algebra, there is a fibrewise morphism

$$\begin{aligned} \gamma : \mathcal{AC} \times_X S &\xrightarrow{X} S, \\ (P \times (\mathbb{CC}\ell(n) \times \Psi(n))) / GL(2^{n/2}, \mathbb{C}) &\rightarrow \\ (P \times \gamma(\mathbb{CC}\ell(n) \times \Psi(n))) / GL(2^{n/2}, \mathbb{C}), \end{aligned} \quad (6.5)$$

of the  $P$ -associated fibre bundles  $\mathcal{AC}$  (6.3) and  $S$  (6.4) with a structure group  $GL(2^{n/2}, \mathbb{C})$ .

**Remark 6.2:** Let  $X$  be a smooth real manifold of dimension  $2^{n/2}$ ,  $n = 2, 4, \dots$ . Let  $TX$  be the tangent bundle over  $X$  and  $LX$  the associated principal frame bundle. Their structure group is  $GL(2^{n/2}, \mathbb{R})$ . There is the canonical group monomorphism

$$GL(2^{n/2}, \mathbb{R}) \rightarrow GL(2^{n/2}, \mathbb{C}). \quad (6.6)$$

Let us consider a trivial complex line bundle  $X \times \mathbb{C}$  over  $X$  and a complexification

$$\mathbb{C}TX = (X \times \mathbb{C}) \otimes_X TX = \mathbb{C} \otimes_X TX \quad (6.7)$$

of  $TX$ . This is a fibre bundle with a structure group  $GL(2^{n/2}, \mathbb{C})$  which is reducible to its subgroup  $GL(2^{n/2}, \mathbb{R})$  (6.6). Let  $P \rightarrow X$  be an associated principal  $GL(2^{n/2}, \mathbb{C})$ -bundle. There is the corresponding monomorphism of principal bundles

$$LX \xrightarrow{X} P. \quad (6.8)$$

Let  $\mathcal{C}$  be the  $P$ -associated bundle (6.1) in complex Clifford algebras  $\mathbb{CC}\ell(n)$ . Let  $S$  (6.4) be a spinor subbundle of  $\mathcal{C}$  whose typical fibre  $\mathbb{C}^{2^{n/2}}$  carries out the natural matrix representation of  $GL(2^{n/2}, \mathbb{C})$ . Due to the monomorphism (6.8), its structure group is reducible to  $GL(2^{n/2}, \mathbb{R})$ , and  $S$  is a  $LX$ -associated bundle isomorphic to the complex tangent bundle  $\mathbb{C}TX$  (6.7). Thus, a complexification of the tangent bundle of a smooth real manifold of dimension  $2^{n/2}$ ,  $n = 2, 4, \dots$  can be represented as a spinor bundle. Moreover, general covariant transformations of  $LX$  gives rise to principal automorphisms of a principal bundle  $P$  (Remark 5.7). Therefore, a principal bundle  $P$ , a  $P$ -associated Clifford algebra bundle  $\mathcal{C}$  and its spinor subbundle  $S$  are natural bundles (Remark 1.1).  $\square$

It should be emphasized that, though there is the ring monomorphism  $\mathcal{C}\ell(m, n-m) \rightarrow \mathbb{CC}\ell(n)$  (2.38), the Clifford algebra bundle  $\mathcal{C}$  (6.1) need not contains a subbundle in real Clifford algebras  $\mathcal{C}\ell(m, n-m)$ , unless a structure group  $GL(2^{n/2}, \mathbb{C})$  of  $\mathcal{C}$  is reducible to a group  $\mathcal{GC}\ell(m, n-m)$ . This problem can be solved as follows.

Let  $X$  be a smooth real manifold of even dimension  $n$ . Let  $TX$  be the tangent bundle over  $X$  and  $LX$  the associated principal frame bundle. Let us assume that their structure group is  $GL(n, \mathbb{R})$  is reducible to a pseudo-orthogonal subgroup  $O(m, n-m)$ . In accordance with Theorem 5.5, a structure group

$GL(n, \mathbb{R})$  always is reducible to a subgroup  $O(n, \mathbb{R})$ . There is the exact sequence of groups (3.19):

$$e \rightarrow \mathbb{Z}_2 \longrightarrow \text{Pin}(m, n-m) \xrightarrow{\zeta} O(m, n-m) \rightarrow e. \quad (6.9)$$

A problem is that this exact sequence is not split, i.e., there is no monomorphism  $\kappa : O(m, n-m) \rightarrow \text{Pin}(m, n-m)$  so that  $\zeta \circ \kappa = \text{Id}$  (Example 3.2 and Remark 6.3).

In this case, we say that a principal  $\text{Pin}(m, n-m)$ -bundle  $\tilde{P}^h \rightarrow X$  is an extension of a principal  $O(m, n-m)$ -bundle  $P^h \rightarrow X$  if there is an epimorphism of principal bundles

$$\tilde{P}^h \rightarrow P^h \quad (6.10)$$

(Remark 5.3). Such an extension need not exist. The following is a corollary of the well-known theorem [4, 5, 9].

**THEOREM 6.1:** The topological obstruction to that a principal  $O(m, n-m)$ -bundle  $P^h \rightarrow X$  lifts to a principal  $\text{Pin}(m, n-m)$ -bundle  $\tilde{P}^h \rightarrow X$  is given by the Čech cohomology group  $H^2(X; \mathbb{Z}_2)$  of  $X$ . Namely, a principal bundle  $P$  defines an element of  $H^2(X; \mathbb{Z}_2)$  which must be zero so that  $P^h \rightarrow X$  can give rise to  $\tilde{P}^h \rightarrow X$ . Inequivalent lifts of  $P^h \rightarrow X$  to principal  $\text{Pin}(m, n-m)$ -bundles are classified by elements of the Čech cohomology group  $H^1(X; \mathbb{Z}_2)$ .  $\square$

Let  $L^h X$  be a reduced principal  $O(m, n-m)$ -subbundle of a frame bundle. In this case, the topological obstruction in Theorem 6.1 to that this bundle  $L^h X$  is extended to a principal  $\text{Pin}(m, n-m)$ -bundle  $\tilde{L}^h X$  is the second Stiefel–Whitney class  $w_2(X) \in H^2(X; \mathbb{Z}_2)$  of  $X$  [9]. Let us assume that a manifold  $X$  is orientable, i.e., the Čech cohomology group  $H^1(X; \mathbb{Z}_2)$  is trivial, and that the second Stiefel–Whitney class  $w_2(X) \in H^2(X; \mathbb{Z}_2)$  of  $X$  also is trivial. Let (6.10) be a desired  $\text{Pin}(m, n-m)$ -lift of a principal  $O(m, n-m)$ -bundle  $P^h$ .

Owing to the canonical monomorphism (2.38) of Clifford algebras, there is the group monomorphism  $\text{Pin}(m, n-m) \rightarrow \mathcal{GCCl}(n)$ . Due to this monomorphism, there exists a principal  $\mathcal{GCCl}(n)$ -bundle  $P$  whose reduced  $\text{Pin}(m, n-m)$ -subbundle is  $\tilde{L}^h X$ , and whose structure group  $\mathcal{GCCl}(n)$  thus is reducible to  $\text{Pin}(m, n-m)$ . Let

$$\mathcal{C}^h \rightarrow X \quad (6.11)$$

be the  $P$ -associated Clifford algebra bundle (6.1). Then it contains a subbundle

$$\mathcal{C}^h(m, n-m) \rightarrow X \quad (6.12)$$

in real Clifford algebras  $\mathcal{Cl}(m, n-m)$ . This subbundle in turn contains a subbundle of generating vector spaces which is  $L^h X$ -associated and, thus, isomorphic to the tangent bundle  $TX$  as a  $L^h X$ -associated bundle. The Clifford algebra bundle  $\mathcal{C}^h$  (6.11) contains spinor subbundles  $S^h \rightarrow X$  (6.4) together with the representation morphisms (6.5).

Of course, with a different reduced principal  $O(m, n-m)$ -subbundle  $L^{h'} X$  of  $LX$ , we come to a different Clifford algebra bundle  $\mathcal{C}^{h'}$  (6.11). Let us recall

that, in accordance with Theorem 5.2, there is one-to-one correspondence (5.20) between the reduced principal  $O(m, n-m)$ -subbundle  $L^h X$  of  $LX$  and the global sections of the quotient bundle

$$\Sigma(m, n-m) = LX/O(m, n-m) \rightarrow X \quad (6.13)$$

which are pseudo-Riemannian metrics on  $X$  of signature  $(m, n-m)$ .

**Remark 6.3:** Let  $X$  be a four-dimensional manifold. In this case, we have the fibre bundle  $\mathcal{C}$  in Clifford algebras  $\mathbb{C}\mathcal{C}\ell(4)$  in Example 6.2 and the Clifford algebra bundles  $\mathcal{C}^h$  (6.11) in  $\mathbb{C}\mathcal{C}\ell(4)$ . A difference between them lies in the fact that a structure group  $GL(4, \mathbb{C})$  of  $\mathcal{C}$  is reducible to a subgroup  $O(m, 4-m)$ , whereas the a structure group of  $\mathcal{C}^h$  is reducible to  $\text{Pin}(m, 4-m)$ , but  $O(m, 4-m)$  is not a subgroup of  $\text{Pin}(m, 4-m)$ .  $\square$

A key point is that, given different sections  $h$  and  $h'$  of the quotient bundle  $\Sigma(m, n-m)$  (6.13), the Clifford algebra bundles  $\mathcal{C}^h$  and  $\mathcal{C}^{h'}$  need not be isomorphic as follows. These fibre bundles are associated to principal  $\text{Pin}(m, n-m)$ -bundles  $\tilde{L}^h X$  and  $\tilde{L}^{h'} X$  which are the two-fold covers (6.10) of the reduced principal  $O(m, n-m)$ -subbundles  $L^h X$  and  $L^{h'} X$  of a frame bundle  $LX$ , respectively. These subbundles need not be isomorphic, and then the principal bundles  $\tilde{L}^h X$ ,  $\tilde{L}^{h'} X$  and, consequently, associated Clifford algebra bundles  $\mathcal{C}^h$ ,  $\mathcal{C}^{h'}$  (6.11) are well. Moreover, let principal bundles  $L^h X$  and  $L^{h'} X$  be isomorphic. For instance, in accordance with Theorem 5.7, this is the case of an orthogonal group  $O(n, o) = O(n, \mathbb{R})$ . However, their covers  $\tilde{L}^h X$  and  $\tilde{L}^{h'} X$  need not be isomorphic. Thus a Clifford algebra bundle must be considered only in a pair with a certain pseudo-Riemannian metric  $h$ .

## 6.2 Composite bundles in Clifford algebras

In order to describe a whole family of non-isomorphic Clifford algebra bundles  $\mathcal{C}^h$ , one can follow a construction of the composite bundle (5.28). Let us consider the composite bundle (5.17):

$$LX \rightarrow \Sigma(m, n-m) \rightarrow X \quad (6.14)$$

where

$$LX \xrightarrow{\pi_{P\Sigma}} \Sigma(m, n-m) \quad (6.15)$$

is a principal bundle with a structure group  $O(m, n-m)$ . Let us consider its principal  $\text{Pin}(m, n-m)$ -cover (6.10):

$$\tilde{P}_\Sigma \rightarrow \Sigma(m, n-m) \quad (6.16)$$

if it exists. Then, given a global section  $h$  of  $\Sigma(m, n-m) \rightarrow X$  (6.13), the pull-back  $h^* \tilde{P}_\Sigma$  is a subbundle of  $\tilde{P}_\Sigma \rightarrow X$  which a  $\text{Pin}(m, n-m)$ -cover

$$h^* \tilde{P}_\Sigma \rightarrow L^h X.$$

Owing to the canonical monomorphism (2.38) of Clifford algebras, there is the group monomorphism  $\text{Pin}(m, n - m) \rightarrow \mathcal{GCCl}(n)$ . Due to this monomorphism, there exists a principal  $\mathcal{GCCl}(n)$ -bundle

$$\tilde{L}X_\Sigma \rightarrow \Sigma(m, n - m). \quad (6.17)$$

Let

$$\mathcal{C}_\Sigma \rightarrow \Sigma(m, n - m) \quad (6.18)$$

be the associated Clifford algebra bundle. It contains a subbundle

$$\mathcal{C}_\Sigma(m, n - m) \rightarrow \Sigma(m, n - m) \quad (6.19)$$

and the spinor subbundles

$$S_\Sigma \rightarrow \Sigma(m, n - m). \quad (6.20)$$

**THEOREM 6.2:** Given a global section  $h$  of  $\Sigma(m, n - m) \rightarrow X$  (6.13), the pull-back bundles  $h^*\mathcal{C}_\Sigma \rightarrow X$ ,  $h^*\mathcal{C}_\Sigma(m, n - m) \rightarrow X$  and  $h^*S_\Sigma \rightarrow X$  are subbundles of the composite bundles  $\mathcal{C}_\Sigma \rightarrow X$ ,  $\mathcal{C}_\Sigma(m, n - m) \rightarrow X$  and  $S_\Sigma \rightarrow X$  and are the bundles  $\mathcal{C}^h \rightarrow X$  (6.11),  $\mathcal{C}^h(m, n - m) \rightarrow X$  (6.12) and  $S^h \rightarrow X$ , respectively.  $\square$

As was mentioned above, this is just the case of gravitation theory that we study in forthcoming Part II of our work.

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